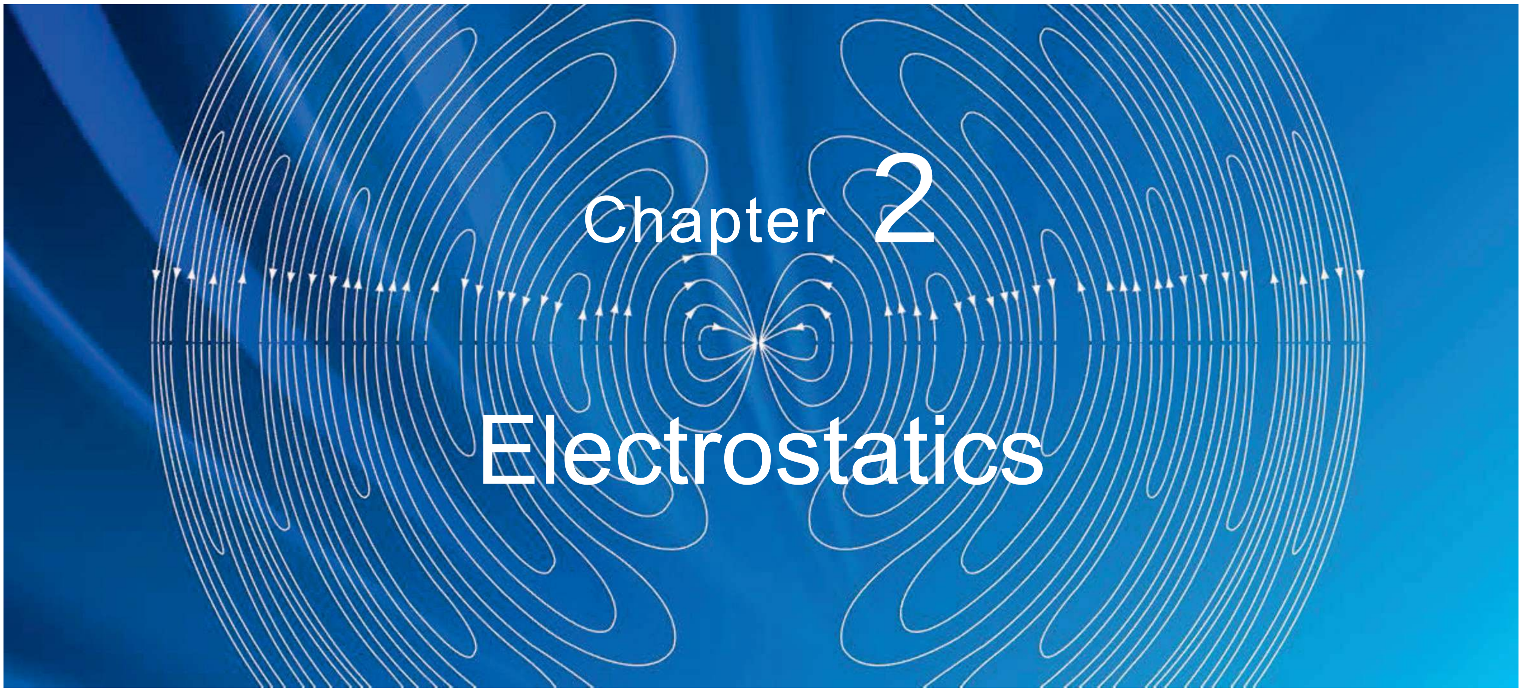

Chapter 2

Electrostatics

Chapter Contents

Objectives

Upon learning the material presented in this chapter, you should be able to:

1. Evaluate the electric field and electric potential due to any distribution of electric charges.
2. Apply Gauss's law.
3. Calculate the resistance *R* of any shaped object given the electric field at every point in its volume.
4. Describe the operational principles of resistive and capacitive sensors.
5. Calculate the capacitance of two-conductor configurations.

2-1 Maxwell's Equations

The modern theory of electromagnetism is based on a set of four fundamental relations known as ***Maxwell's equations***:

$$\nabla \cdot \mathbf{D} = \rho_v, \tag{4.1a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4.1b}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4.1c}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \tag{4.1d}$$

Here $\mathbf{E}$ and $\mathbf{D}$ are the ***electric field intensity*** and ***flux density*** interrelated by $\mathbf{D} = \varepsilon\mathbf{E}$ where ε is the electrical permittivity; $\mathbf{H}$ and $\mathbf{B}$ are ***magnetic field intensity*** and ***flux density*** interrelated by $\mathbf{B} = \mu\mathbf{H}$ where μ is the magnetic permeability; ρ_v is the electric charge density per unit volume; and $\mathbf{J}$ is the current density per unit area. The fields and fluxes $\mathbf{E}$, $\mathbf{D}$, $\mathbf{B}$, $\mathbf{H}$ were introduced in Section 1-3, and ρ_v and $\mathbf{J}$ will be discussed in Section 4-2. Maxwell's equations hold in any material, including free space (vacuum). In general, all of the above quantities may depend on spatial location and time t. In the interest of readability, we will not, however, explicitly reference these dependencies (as in $\mathbf{E}(x,y,z,t)$) except when the context calls for it. By formulating these equations, published in a classic treatise in 1873, James Clerk Maxwell established the first unified theory of electricity and magnetism. His equations are deduced from experimental observations reported by Coulomb, Gauss, Ampère, Faraday, and others; they not only encapsulate the connection between the electric field and electric charge and between the magnetic field and electric current but also capture the bilateral coupling between electric and magnetic fields and fluxes. Together with some auxiliary relations, Maxwell's equations comprise the fundamental tenets of electromagnetic theory.

▶ Under ***static*** conditions, none of the quantities appearing in Maxwell's equations are functions of time (i.e., $\partial/\partial t = 0$). *This happens when all charges are permanently fixed in space. If they move, they do so at a steady rate so that ρ_v and $\mathbf{J}$ are constant in time.* ◀

Under these circumstances, the time derivatives of $\mathbf{B}$ and $\mathbf{D}$ in Eqs. (4.1b) and (4.1d) vanish, and Maxwell's equations reduce to the following pairs.

Electrostatics

$$\nabla \cdot \mathbf{D} = \rho_v, \tag{4.2a}$$

$$\nabla \times \mathbf{E} = 0. \tag{4.2b}$$

Magnetostatics

$$\nabla \cdot \mathbf{B} = 0, \tag{4.3a}$$

$$\nabla \times \mathbf{H} = \mathbf{J}. \tag{4.3b}$$

Maxwell's four equations separate into two uncoupled pairs with the first pair involving only the electric field and flux $\mathbf{E}$ and $\mathbf{D}$ and the second pair involving only the magnetic field and flux $\mathbf{H}$ and $\mathbf{B}$.

▶ Electric and magnetic fields become ***decoupled*** under static conditions. ◀

This allows us to study electricity and magnetism as two distinct and separate phenomena as long as the spatial distributions of charge and current flow remain constant in time. We refer to the study of electric and magnetic phenomena under static conditions as ***electrostatics*** and ***magnetostatics***, respectively. Electrostatics is the subject of the present chapter, and we learn about magnetostatics in Chapter 5. The experience gained through studying electrostatic and magnetostatic phenomena will prove invaluable in tackling the more involved material in subsequent chapters that deal with time-varying fields, charge densities, and currents.

We study electrostatics not only as a prelude to the study of time-varying fields but also because it is an important field in its own right. Many electronic devices and systems are based on the principles of electrostatics. They include x-ray machines, oscilloscopes, ink-jet electrostatic printers, liquid crystal displays, copy machines, micro-electromechanical switches and accelerometers, and many solid-state–based control devices. Electrostatic principles also guide the design of medical diagnostic sensors, such as the electrocardiogram, which records the heart's pumping pattern, and the electroencephalogram, which records brain activity, as well as the development of numerous industrial applications.

2-2 Charge and Current Distributions

In electromagnetics, we encounter various forms of electric charge distributions. When put in motion, these charge distributions constitute current distributions. Charges and currents may be distributed over a volume of space, across a surface, or along a line.

2-2.1 Charge Densities

At the atomic scale, the charge distribution in a material is discrete, meaning that charge exists only where electrons and nuclei are and nowhere else. In electromagnetics, we usually are interested in studying phenomena at a much larger scale, typically three or more orders of magnitude greater than the spacing between adjacent atoms. At such a macroscopic scale, we can disregard the discontinuous nature of the charge distribution and treat the net charge contained in an elemental volume $\Delta \mathcal{V}$ as if it were uniformly distributed within. Accordingly, we define the ***volume charge density*** ρ_v as

$$\rho_v = \lim_{\Delta \mathcal{V} \to 0} \frac{\Delta q}{\Delta \mathcal{V}} = \frac{dq}{d\mathcal{V}} \qquad (\text{C/m}^3), \tag{4.4}$$

where Δq is the charge contained in $\Delta \mathcal{V}$. In general, ρ_v depends on spatial location (x, y, z) and t; thus, $\rho_v = \rho_v(x, y, z, t)$. Physically, ρ_v represents the average charge per unit volume for a volume $\Delta \mathcal{V}$ centered at (x, y, z) with $\Delta \mathcal{V}$ being large enough to contain a large number of atoms, yet it is small enough to be regarded as a point at the macroscopic scale under consideration. The variation of ρ_v with spatial location is called its ***spatial distribution*** (or simply its ***distribution***). The total charge contained in volume $\mathcal{V}$ is

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} \qquad (\text{C}). \tag{4.5}$$

In some cases, particularly when dealing with conductors, electric charge may be distributed across the surface of a material, where the quantity of interest is the ***surface charge density*** ρ_s, which is defined as

$$\rho_s = \lim_{\Delta s \to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \qquad (\text{C/m}^2), \tag{4.6}$$

where Δq is the charge present across an elemental surface area Δs. Similarly, if the charge is, for all practical purposes, confined to a line, which need not be straight, we characterize its distribution in terms of the ***line charge density*** ρ_ℓ, defined as

$$\rho_\ell = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \qquad (\text{C/m}). \tag{4.7}$$

Example2-1: Line Charge Distribution

Calculate the total charge Q contained in a cylindrical tube oriented along the z axis, as shown in **Fig. 4-1(a)**. The line charge density is $\rho_\ell = 2z$, where z is the distance in meters from the bottom end of the tube. The tube length is 10 cm.

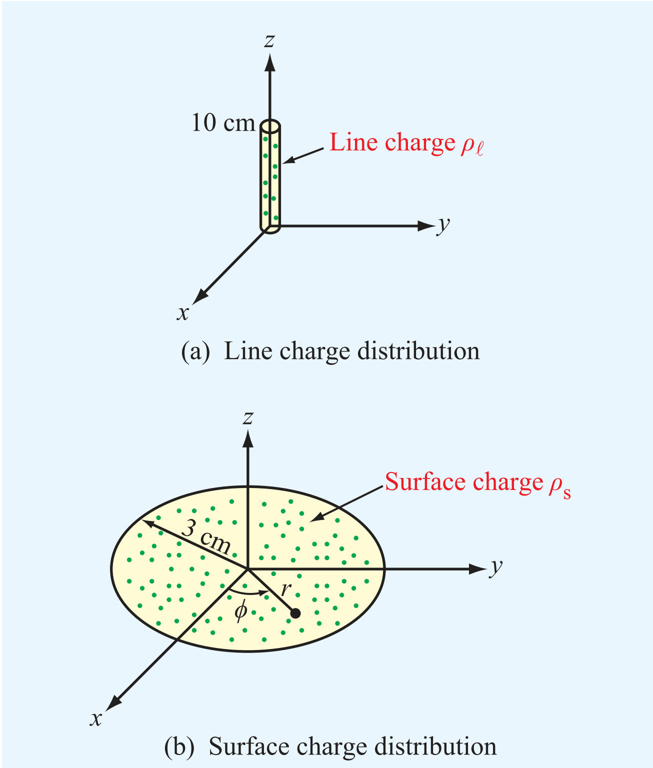

Figure 4-1 Charge distributions for Examples 4-1 and 4-2.

Solution: The total charge Q is

$$Q = \int_0^{0.1} \rho_\ell \, dz = \int_0^{0.1} 2z \, dz = z^2 \Big|_0^{0.1} = 10^{-2} \text{ C}.$$

Example2-2: Surface Charge Distribution

The circular disk of electric charge shown in **Fig. 4-1(b)** is characterized by an azimuthally symmetric surface charge density that increases linearly with r from zero at the center to 6 C/m^2 at $r = 3$ cm. Find the total charge present on the disk surface.

Solution: Since ρ_s is symmetrical with respect to the azimuth angle ϕ, it depends only on r and is given by

$$\rho_s = \frac{6r}{3 \times 10^{-2}} = 2 \times 10^2 r \qquad (\text{C/m}^2),$$

where r is in meters. In polar coordinates, an elemental area is $ds = r \, dr \, d\phi$, and for the disk shown in **Fig. 4-1(b)**, the limits of integration are from 0 to 2π (rad) for ϕ and from 0 to 3×10^{-2} m for r. Hence,

$$Q = \int_S \rho_s \, ds = \int_{\phi=0}^{2\pi} \int_{r=0}^{3\times10^{-2}} (2 \times 10^2 r) r \, dr \, d\phi$$

$$= 2\pi \times 2 \times 10^2 \left. \frac{r^3}{3} \right|_0^{3\times10^{-2}} = 11.31 \quad \text{(mC)}.$$

Exercise 4-1: A square plate residing in the x–y plane is situated in the space defined by $-3 \text{ m} \le x \le 3$ m and $-3 \text{ m} \le y \le 3$ m. Find the total charge on the plate if the surface charge density is $\rho_s = 4y^2$ (μC/m^2).

Answer: $Q = 0.432$ (mC). (See EM.)

Exercise 4-2: A thick spherical shell centered at the origin extends between $R = 2$ cm and $R = 3$ cm. If the volume charge density is $\rho_v = 3R \times 10^{-4}$ (C/m^3), find the total charge contained in the shell.

Answer: $Q = 0.61$ (nC). (See EM.)

2-2.2 Current Density

Consider a tube with volume charge density ρ_v (**Fig. 4-2(a)**). The charges in the tube move with velocity $\mathbf{u}$ along the tube axis. Over a period Δt, the charges move a distance $\Delta l = u \, \Delta t$. The amount of charge that crosses the tube's cross-sectional surface $\Delta s'$ in time Δt is therefore

$$\Delta q' = \rho_v \, \Delta \mathcal{V} = \rho_v \, \Delta l \, \Delta s' = \rho_v u \, \Delta s' \, \Delta t. \tag{4.8}$$

Now consider the more general case where the charges are flowing through a surface Δs with normal $\hat{\mathbf{n}}$ not necessarily parallel to $\mathbf{u}$ (**Fig. 4-2(b)**). In this case, the amount of charge Δq flowing through Δs is

$$\Delta q = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} \, \Delta t, \tag{4.9}$$

where $\Delta \mathbf{s} = \hat{\mathbf{n}} \, \Delta s$ and the corresponding total current flowing in the tube is

$$\Delta I = \frac{\Delta q}{\Delta t} = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} = \mathbf{J} \cdot \Delta \mathbf{s}. \tag{4.10}$$

Here

$$\mathbf{J} = \rho_v \mathbf{u} \qquad (\text{A/m}^2) \tag{4.11}$$

is defined as the ***current density*** in ampere per square meter. Generalizing to an arbitrary surface S, the total current flowing through it is

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \qquad \text{(A)}. \tag{4.12}$$

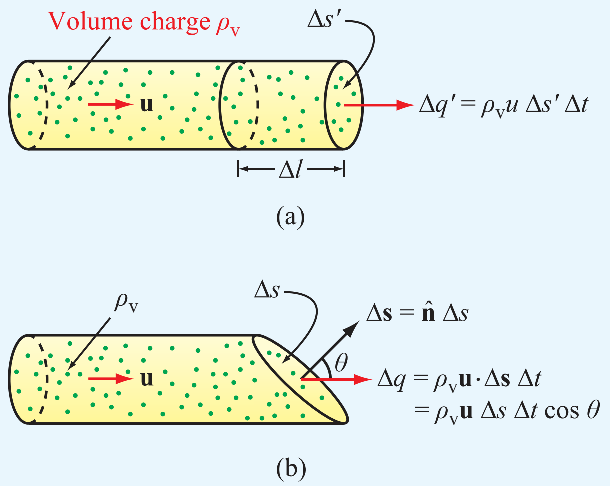

Figure 4-2 Charges with velocity $\mathbf{u}$ moving through a cross section $\Delta s'$ in (a) and Δs in (b).

▶ When a current is due to the actual movement of electrically charged matter, it is called a ***convection current***, and $\mathbf{J}$ is called a ***convection current density***. ◀

A wind-driven charged cloud, for example, gives rise to a convection current. In some cases, the charged matter constituting the convection current consists solely of charged particles, such as the electron beam of a scanning electron microscope or the ion beam of a plasma propulsion system.

When a current is due to the movement of charged particles relative to their host material, $\mathbf{J}$ is called a ***conduction current density***. In a metal wire, for example, there are equal amounts of positive charges (in atomic nuclei) and negative charges (in the electron shells of the atoms). None of the positive charges and few of the negative charges can move; only those electrons in the outermost electron shells of the atoms can be pushed from one atom to the next if a voltage is applied across the ends of the wire.

▶ This movement of electrons from atom to atom constitutes a ***conduction current***. The electrons that emerge from the wire are not necessarily the same electrons that entered the wire at the other end. ◀

Conduction current, which is discussed in more detail in Section2-6, obeys Ohm's law, whereas convection current does not.

Concept Question 2-1: What happens to Maxwell's equations under static conditions?

Concept Question 2-2: How is the current density $\mathbf{J}$ related to the volume charge density ρ_v?

Concept Question 2-3: What is the difference between convection and conduction currents?

2-3 Coulomb's Law

One of the primary goals of this chapter is to develop dexterity in applying the expressions for the ***electric field intensity*** $\mathbf{E}$ and associated ***electric flux density*** $\mathbf{D}$ induced by a specified distribution of charge. Our discussion will be limited to electrostatic fields induced by stationary charge densities.

We begin by reviewing the expression for the electric field introduced in Section 1-3.2 on the basis of the results of Coulomb's experiments on the electrical force between charged bodies. ***Coulomb's law***, which was first introduced for electrical charges in air and later generalized to material media, implies that:

(1) An isolated charge q induces an electric field $\mathbf{E}$ at every point in space, and at any specific point P, $\mathbf{E}$ is given by

$$\mathbf{E} = \hat{\mathbf{R}}\,\frac{q}{4\pi\varepsilon R^2} \qquad \text{(V/m)}, \tag{4.13}$$

where $\hat{\mathbf{R}}$ is a unit vector pointing from q to P (**Fig. 4-3**), R is the distance between them, and ε is the electrical permittivity of the medium containing the observation point P.

(2) In the presence of an electric field $\mathbf{E}$ at a given point in space that may be due to a single charge or a distribution of charges, the force acting on a test charge q' when placed at P is

$$\mathbf{F} = q'\mathbf{E} \qquad \text{(N)}. \tag{4.14}$$

With $\mathbf{F}$ measured in newtons (N) and q' in coulombs (C), the unit of $\mathbf{E}$ is (N/C), which will be shown later in Section 4-5 to be the same as volt per meter (V/m).

For a material with electrical permittivity ε, the electric field quantities $\mathbf{D}$ and $\mathbf{E}$ are related by

$$\mathbf{D} = \varepsilon\mathbf{E} \tag{4.15}$$

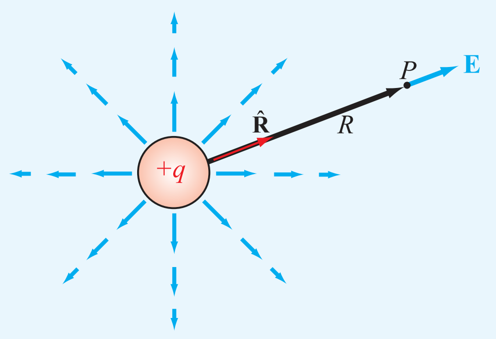

Figure 4-3 Electric field lines due to a charge q.

with

$$\varepsilon = \varepsilon_\mathrm{r}\varepsilon_0, \tag{4.16}$$

where

$$\varepsilon_0 = 8.85 \times 10^{-12} \approx (1/36\pi) \times 10^{-9} \qquad \text{(F/m)}$$

is the electrical permittivity of free space and $\varepsilon_\mathrm{r} = \varepsilon/\varepsilon_0$ is called the ***relative permittivity*** (or ***dielectric constant***) of the material. For most materials and under a wide range of conditions, ε is independent of both the magnitude and direction of $\mathbf{E}$ [as implied by Eq. (4.15)].

▶ If ε is independent of the magnitude of $\mathbf{E}$, then the material is said to be ***linear*** because $\mathbf{D}$ and $\mathbf{E}$ are related linearly, and if it is independent of the direction of $\mathbf{E}$, the material is said to be ***isotropic.*** ◀

Materials usually do not exhibit nonlinear permittivity behavior except when the amplitude of $\mathbf{E}$ is very high (at levels approaching dielectric breakdown conditions discussed later in Section 4-7), and anisotropy is present only in certain materials with peculiar crystalline structures. Hence, except for unique materials under very special circumstances, the quantities $\mathbf{D}$ and $\mathbf{E}$ are effectively redundant; for a material with known ε, knowledge of either $\mathbf{D}$ or $\mathbf{E}$ is sufficient to specify the other in that material.

2-3.1 Electric Field Due to Multiple Point Charges

The expression given by Eq. (4.13) for the field $\mathbf{E}$ due to a single point charge can be extended to multiple charges. We begin by considering two point charges, q_1 and q_2, with position vectors $\mathbf{R}_1$ and $\mathbf{R}_2$ (measured from the origin in

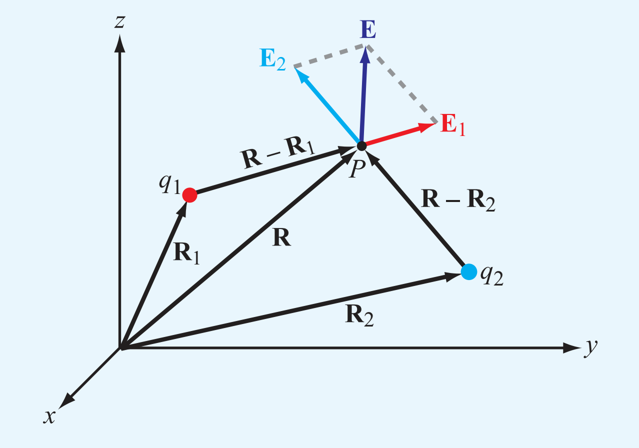

Figure 2-4 The electric field $\mathbf{E}$ at P due to two charges is equal to the vector sum of $\mathbf{E}_1$ and $\mathbf{E}_2$.

Fig.2-4). The electric field $\mathbf{E}$ is to be evaluated at a point P with position vector $\mathbf{R}$. At P, the electric field $\mathbf{E}_1$ due to q_1 alone is given by Eq. (2.13) with$R$, which is the distance between q_1 and P, replaced with $|\mathbf{R}-\mathbf{R}_1|$ and the unit vector $\hat{\mathbf{R}}$ replaced with $(\mathbf{R}-\mathbf{R}_1)/|\mathbf{R}-\mathbf{R}_1|$. Thus,

$$\mathbf{E}_1 = \frac{q_1(\mathbf{R}-\mathbf{R}_1)}{4\pi\varepsilon|\mathbf{R}-\mathbf{R}_1|^3} \quad \text{(V/m)}. \tag{4.17a}$$

Similarly, the electric field at P due to q_2 alone is

$$\mathbf{E}_2 = \frac{q_2(\mathbf{R}-\mathbf{R}_2)}{4\pi\varepsilon|\mathbf{R}-\mathbf{R}_2|^3} \quad \text{(V/m)}. \tag{4.17b}$$

▶ The electric field obeys the principle of linear superposition. ◀

Hence, the total electric field $\mathbf{E}$ at P due to q_1 and q_2 together is determined as

$$\begin{aligned}\mathbf{E} &= \mathbf{E}_1 + \mathbf{E}_2 \\ &= \frac{1}{4\pi\varepsilon}\left[\frac{q_1(\mathbf{R}-\mathbf{R}_1)}{|\mathbf{R}-\mathbf{R}_1|^3} + \frac{q_2(\mathbf{R}-\mathbf{R}_2)}{|\mathbf{R}-\mathbf{R}_2|^3}\right].\end{aligned} \tag{4.18}$$

Generalizing the preceding result to the case of N point charges, the electric field $\mathbf{E}$ at point P with position vector $\mathbf{R}$ due to charges $q_1, q_2, \ldots, q_N$ located at points with position vectors $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_N$ equals the vector sum of the electric fields induced by all the individual charges. Thus,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i(\mathbf{R}-\mathbf{R}_i)}{|\mathbf{R}-\mathbf{R}_i|^3} \quad \text{(V/m)}. \tag{4.19}$$

Example 2-3: Electric Field Due to Two Point Charges

Two point charges with

$$q_1 = 2\times10^{-5}\ \text{C}$$

and

$$q_2 = -4\times10^{-5}\ \text{C}$$

are located in free space at points with Cartesian coordinates $(1,3,-1)$ and $(-3,1,-2)$, respectively. Find (a) the electric field $\mathbf{E}$ at $(3,1,-2)$ and (b) the force on a 8×10^{-5} C charge located at that point. All distances are in meters.

Solution: (a) From Eq. (4.18), the electric field $\mathbf{E}$ with $\varepsilon=\varepsilon_0$ (free space) is

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left[q_1\frac{(\mathbf{R}-\mathbf{R}_1)}{|\mathbf{R}-\mathbf{R}_1|^3} + q_2\frac{(\mathbf{R}-\mathbf{R}_2)}{|\mathbf{R}-\mathbf{R}_2|^3}\right] \quad \text{(V/m)}.$$

The vectors $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}$ are

$$\begin{aligned}\mathbf{R}_1 &= \hat{\mathbf{x}} + \hat{\mathbf{y}}3 - \hat{\mathbf{z}}, \\ \mathbf{R}_2 &= -\hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2, \\ \mathbf{R} &= \hat{\mathbf{x}}3 + \hat{\mathbf{y}} - \hat{\mathbf{z}}2.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbf{E} &= \frac{1}{4\pi\varepsilon_0}\left[\frac{2(\hat{\mathbf{x}}2-\hat{\mathbf{y}}2-\hat{\mathbf{z}})}{27} - \frac{4(\hat{\mathbf{x}}6)}{216}\right]\times10^{-5} \\ &= \frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}4-\hat{\mathbf{z}}2}{108\pi\varepsilon_0}\times10^{-5} \quad \text{(V/m)}.\end{aligned}$$

(b) The force on q_3 is

$$\begin{aligned}\mathbf{F} = q_3\mathbf{E} &= 8\times10^{-5}\times\frac{\hat{\mathbf{x}}-\hat{\mathbf{y}}4-\hat{\mathbf{z}}2}{108\pi\varepsilon_0}\times10^{-5} \\ &= \frac{\hat{\mathbf{x}}2-\hat{\mathbf{y}}8-\hat{\mathbf{z}}4}{27\pi\varepsilon_0}\times10^{-10} \quad \text{(N)}.\end{aligned}$$

Exercise2-3: Four charges of 10 μC each are located in free space at points with Cartesian coordinates $(-3,0,0)$, $(3,0,0)$, $(0,-3,0)$, and $(0,3,0)$. Find the force on a 20 μC charge located at $(0,0,4)$. All distances are in meters.

Answer: $\mathbf{F} = \hat{\mathbf{z}}0.23$ N. (See EM.)

Exercise2-4: Two identical charges are located on the x axis at $x=3$ and $x=7$. At what point in space is the net electric field zero?

Answer: At point $(5,0,0)$. (See EM.)

Exercise2-5: In a hydrogen atom, the electron and proton are separated by an average distance of 5.3×10^{-11} m. Find the magnitude of the electrical force F_e between the two particles, and compare it with the gravitational force F_g between them.

Answer: $F_e = 8.2 \times 10^{-8}$ N, and $F_g = 3.6 \times 10^{-47}$ N. (See EM.)

2-3.2 Electric Field Due to a Charge Distribution

We now extend the results obtained for the field due to discrete point charges to continuous charge distributions. Consider a volume $\mathcal{V}'$ that contains a distribution of electric charge with volume charge density ρ_v, which may vary spatially within $\mathcal{V}'$ (**Fig. 4-5**). The differential electric field at a point P due to a differential amount of charge $dq = \rho_v \, d\mathcal{V}'$ contained in a differential volume $d\mathcal{V}'$ is

$$d\mathbf{E} = \hat{\mathbf{R}}' \frac{dq}{4\pi\varepsilon R'^2} = \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{4\pi\varepsilon R'^2}, \tag{4.20}$$

where $\mathbf{R}'$ is the vector from the differential volume $d\mathcal{V}'$ to point P. Applying the principle of linear superposition, the total electric field $\mathbf{E}$ is obtained by integrating the fields due to all differential charges in $\mathcal{V}'$. Thus,

$$\mathbf{E} = \int_{\mathcal{V}'} d\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{R'^2}. \tag{4.21a}$$

(volume distribution)

It is important to note that, in general, both R' and $\hat{\mathbf{R}}'$ vary as a function of position over the integration volume $\mathcal{V}'$.

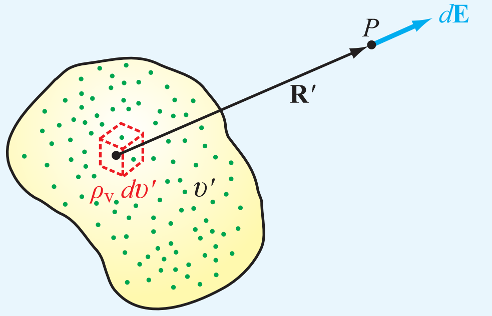

Figure 4-5 Electric field due to a volume charge distribution.

If the charge is distributed across a surface S' with surface charge density ρ_s, then $dq = \rho_s \, ds'$, and if it is distributed along a line l' with a line charge density ρ_ℓ, then $dq = \rho_\ell \, dl'$. Accordingly, the electric fields due to surface and line charge distributions are

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2}, \tag{4.21b}$$

(surface distribution)

$$\mathbf{E} = \frac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2}. \tag{4.21c}$$

(line distribution)

Example2-4: Electric Field of a Ring of Charge

A ring of charge of radius b is characterized by a uniform line charge density of positive polarity ρ_ℓ. The ring resides in free space and is positioned in the x–y plane, as shown in **Fig. 4-6**. Determine the electric field intensity $\mathbf{E}$ at a point $P = (0,0,h)$ along the axis of the ring at a distance h from its center.

Solution: We start by considering the electric field generated by a differential ring segment with cylindrical coordinates $(b,\phi,0)$ in **Fig. 4-6(a)**. The segment has length $dl = b\,d\phi$ and contains charge $dq = \rho_\ell \, dl = \rho_\ell b \, d\phi$. The distance vector $\mathbf{R}'_1$ from segment 1 to point $P = (0,0,h)$ is

$$\mathbf{R}'_1 = -\hat{\mathbf{r}}b + \hat{\mathbf{z}}h,$$

from which it follows that

$$R'_1 = |\mathbf{R}'_1| = \sqrt{b^2+h^2}, \qquad \hat{\mathbf{R}}'_1 = \frac{\mathbf{R}'_1}{|\mathbf{R}'_1|} = \frac{-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h}{\sqrt{b^2+h^2}}.$$

The electric field at $P = (0,0,h)$ due to the charge in segment 1 therefore is

$$d\mathbf{E}_1 = \frac{1}{4\pi\varepsilon_0} \hat{\mathbf{R}}'_1 \frac{\rho_\ell \, dl}{{R'_1}^2} = \frac{\rho_\ell b}{4\pi\varepsilon_0} \frac{(-\hat{\mathbf{r}}b + \hat{\mathbf{z}}h)}{(b^2+h^2)^{3/2}} \, d\phi.$$

The field $d\mathbf{E}_1$ has component dE_{1r} along $-\hat{\mathbf{r}}$ and component dE_{1z} along $\hat{\mathbf{z}}$. From symmetry considerations, the field $d\mathbf{E}_2$ generated by differential segment 2 in **Fig. 4-6(b)**, which is located diametrically opposite to segment 1, is identical to $d\mathbf{E}_1$ except that the $\hat{\mathbf{r}}$ component of $d\mathbf{E}_2$ is opposite that

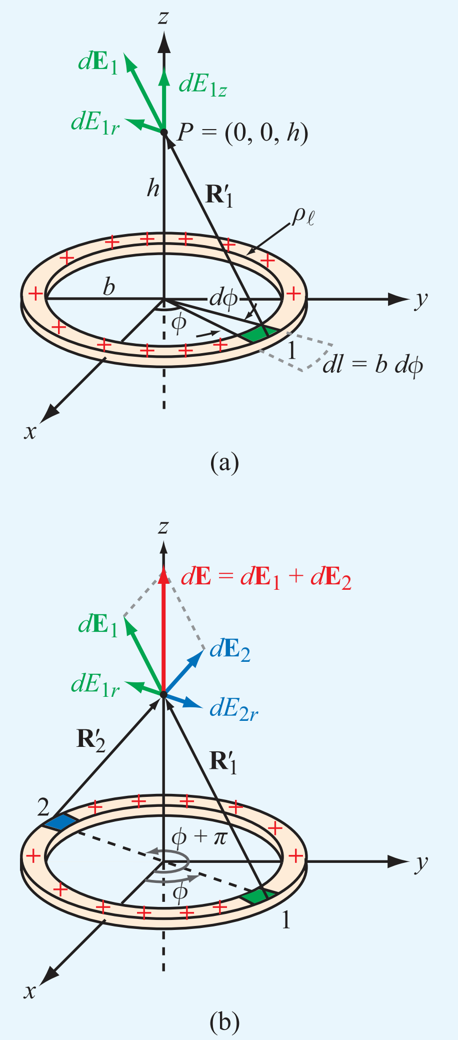

Figure2-6 Ring of charge with line density ρ_ℓ. (a) The field $d\mathbf{E}_1$ due to infinitesimal segment 1 and (b) the fields $d\mathbf{E}_1$ and $d\mathbf{E}_2$ due to segments at diametrically opposite locations (Example 4-4).

of $d\mathbf{E}_1$. Hence, the $\hat{\mathbf{r}}$ components in the sum cancel and the $\hat{\mathbf{z}}$ contributions add. The sum of the two contributions is

$$d\mathbf{E} = d\mathbf{E}_1 + d\mathbf{E}_2 = \hat{\mathbf{z}}\,\frac{\rho_\ell bh}{2\pi\varepsilon_0}\,\frac{d\phi}{(b^2+h^2)^{3/2}}\,. \qquad (4.22)$$

Since for every ring segment in the semicircle defined over the azimuthal range $0 \le \phi \le \pi$ (the right-hand half of the circular ring) there is a corresponding segment located diametrically opposite at $(\phi+\pi)$, we can obtain the total field generated by the ring by integrating Eq. (4.22) over a semicircle as

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{z}}\,\frac{\rho_\ell bh}{2\pi\varepsilon_0(b^2+h^2)^{3/2}}\int_0^\pi d\phi \\ &= \hat{\mathbf{z}}\,\frac{\rho_\ell bh}{2\varepsilon_0(b^2+h^2)^{3/2}} = \hat{\mathbf{z}}\,\frac{h}{4\pi\varepsilon_0(b^2+h^2)^{3/2}}\,Q, \qquad (4.23)\end{aligned}$$

where $Q = 2\pi b\rho_\ell$ is the total charge on the ring.

Example2-5: Electric Field of a Circular Disk of Charge

Find the electric field at point P with Cartesian coordinates $(0,0,h)$ due to a circular disk of radius a and uniform charge density ρ_s residing in the x–y plane (**Fig. 4-7**). Also, evaluate $\mathbf{E}$ due to an infinite sheet of charge density ρ_s by letting $a \to \infty$.

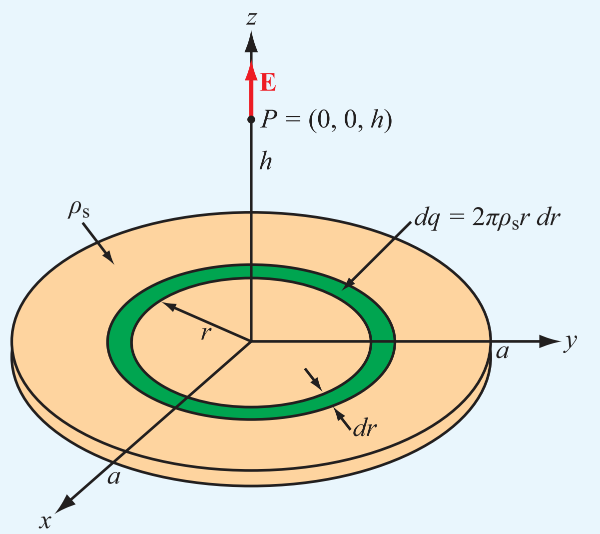

Figure2-7 Circular disk of charge with surface charge density ρ_s. The electric field at $P = (0,0,h)$ points along the z direction (Example 4-5).

Solution: Building on the expression obtained in Example 4-4 for the on-axis electric field due to a circular ring of charge, we can determine the field due to the circular disk by treating the disk as a set of concentric rings. A ring of radius r and width dr has an area $ds = 2\pi r\,dr$ and contains charge $dq = \rho_s\,ds = 2\pi\rho_s r\,dr$. Upon using this expression in

Eq. (4.23) and also replacing b with r, we obtain the following expression for the field due to the ring:

$$d\mathbf{E} = \hat{\mathbf{z}}\,\frac{h}{4\pi\varepsilon_0(r^2+h^2)^{3/2}}\,(2\pi\rho_s r\,dr).$$

The total field at P is obtained by integrating the expression over the limits $r=0$ to $r=a$:

$$\mathbf{E} = \hat{\mathbf{z}}\,\frac{\rho_s h}{2\varepsilon_0}\int_0^a \frac{r\,dr}{(r^2+h^2)^{3/2}} = \pm\hat{\mathbf{z}}\,\frac{\rho_s}{2\varepsilon_0}\left[1-\frac{|h|}{\sqrt{a^2+h^2}}\right], \tag{4.24}$$

with the plus sign for $h>0$ (P above the disk) and the minus sign when $h<0$ (P below the disk).

For an infinite sheet of charge with $a=\infty$,

$$\mathbf{E} = \pm\hat{\mathbf{z}}\,\frac{\rho_s}{2\varepsilon_0}. \tag{4.25}$$

(infinite sheet of charge)

We note that for an infinite sheet of charge $\mathbf{E}$ is the same at all points above the x–y plane, and a similar statement applies for points below the x–y plane.

Concept Question 4-4: When characterizing the electrical permittivity of a material, what do the terms *linear* and *isotropic* mean?

Concept Question 4-5: If the electric field is zero at a given point in space, does this imply the absence of electric charges?

Concept Question 4-6: State the principle of linear superposition as it applies to the electric field due to a distribution of electric charge.

Exercise 4-6: An infinite sheet with uniform surface charge density ρ_s is located at $z=0$ (x–y plane), and another infinite sheet with density $-\rho_s$ is located at $z=2$ m with both in free space. Determine $\mathbf{E}$ everywhere.

Answer: $\mathbf{E}=0$ for $z<0$; $\mathbf{E}=\hat{\mathbf{z}}\rho_s/\varepsilon_0$ for $0<z<2$ m; and $\mathbf{E}=0$ for $z>2$ m. (See EM.)

2-4 Gauss's Law

In this section, we use Maxwell's equations to confirm the expressions for the electric field implied by Coulomb's law, and propose alternative techniques for evaluating electric fields induced by electric charge. To that end, we restate Eq. (4.1a):

$$\nabla\cdot\mathbf{D} = \rho_v, \tag{4.26}$$

(differential form of Gauss's law)

which is referred to as the differential form of ***Gauss's law***. The adjective "differential" refers to the fact that the divergence operation involves spatial derivatives. As we see shortly, Eq. (4.26) can be converted to an integral form. When solving electromagnetic problems, we often go back and forth between equations in differential and integral form, depending on which of the two happens to be the more applicable or convenient to use. To convert Eq. (4.26) into integral form, we multiply both sides by $d\mathcal{V}$ and evaluate their integrals over an arbitrary volume $\mathcal{V}$:

$$\int_{\mathcal{V}} \nabla\cdot\mathbf{D}\,d\mathcal{V} = \int_{\mathcal{V}} \rho_v\,d\mathcal{V} = Q. \tag{4.27}$$

Here, Q is the total charge enclosed in $\mathcal{V}$. The divergence theorem, given by Eq. (3.98), states that the volume integral of the divergence of any vector over a volume $\mathcal{V}$ equals the total outward flux of that vector through the surface S enclosing $\mathcal{V}$. Thus, for the vector $\mathbf{D}$,

$$\int_{\mathcal{V}} \nabla\cdot\mathbf{D}\,d\mathcal{V} = \oint_S \mathbf{D}\cdot d\mathbf{s}. \tag{4.28}$$

Comparison of Eq. (4.27) with Eq. (4.28) leads to

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = Q. \tag{4.29}$$

(integral form of Gauss's law)

► The integral form of Gauss's law is illustrated diagrammatically in **Fig. 4-8**; for each differential surface element $d\mathbf{s}$, $\mathbf{D}\cdot d\mathbf{s}$ is the electric field flux flowing outward of $\mathcal{V}$ through $d\mathbf{s}$, and the total flux through surface S equals the enclosed charge Q. The surface S is called a ***Gaussian surface***. ◄

The integral form of Gauss's law can be applied to determine $\mathbf{D}$ due to a single isolated point charge q by enclosing the latter with a closed, spherical, Gaussian surface S of arbitrary radius R centered at q (**Fig. 4-9**). From symmetry considerations and assuming that q is positive, the direction of $\mathbf{D}$ must be radially outward along the unit vector $\hat{\mathbf{R}}$, and D_R, which is the magnitude of $\mathbf{D}$, must be the same at all points on S. Thus, at any point on S,

$$\mathbf{D} = \hat{\mathbf{R}}D_R, \tag{4.30}$$

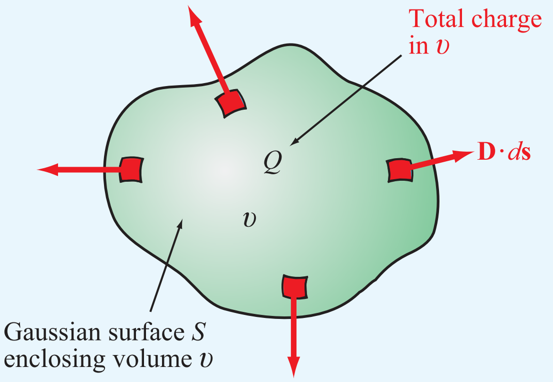

Figure2-8 The integral form of Gauss's law states that the outward flux of **D** through a surface is proportional to the enclosed charge Q.

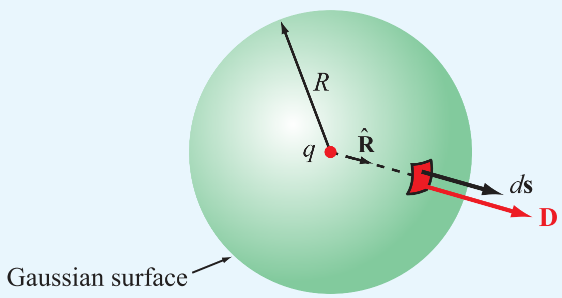

Figure2-9 Electric field **D** due to point charge q.

and $d\mathbf{s} = \hat{\mathbf{R}}\, ds$. Applying Gauss's law gives

$$\oint_S \mathbf{D}\cdot d\mathbf{s} = \oint_S \hat{\mathbf{R}} D_R \cdot \hat{\mathbf{R}}\, ds = \oint_S D_R\, ds = D_R(4\pi R^2) = q. \tag{4.31}$$

Solving for D_R and then inserting the result in Eq. (4.30) gives the expression for the electric field **E** induced by an isolated point charge in a medium with permittivity ε:

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon} = \hat{\mathbf{R}}\,\frac{q}{4\pi\varepsilon R^2} \qquad \text{(V/m)}. \tag{4.32}$$

This is identical with Eq. (4.13) obtained from Coulomb's law; after all, Maxwell's equations incorporate Coulomb's law. For this simple case of an isolated point charge, it does not matter whether Coulomb's law or Gauss's law is used to obtain the expression for **E**. However, it does matter which approach we follow when we deal with multiple point charges or continuous charge distributions. Even though Coulomb's law can be used to find **E** for any specified distribution of charge, Gauss's law is easier to apply than Coulomb's law, but its utility is limited to symmetrical charge distributions.

► Gauss's law, as given by Eq. (4.29), provides a convenient method for determining the flux density **D** when the charge distribution possesses symmetry properties that allow us to infer the variations of the magnitude and direction of **D** as a function of spatial location. This facilitates the integration of **D** over a cleverly chosen Gaussian surface. ◄

Because at every point on the surface the direction of $d\mathbf{s}$ is along its outward normal, only the normal component of **D** at the surface contributes to the integral in Eq. (4.29). To successfully apply Gauss's law, the surface S should be chosen from symmetry considerations so that, across each subsurface of S, **D** is constant in magnitude and its direction is either normal or purely tangential to the subsurface. These aspects are illustrated in Example 4-6.

Example2-6: Electric Field of an Infinite Line Charge

Use Gauss's law to obtain an expression for **E** due to an infinitely long line with uniform charge density ρ_ℓ that resides along the z axis in free space.

Solution: Since the charge density along the line is uniform, infinite in extent, and residing along the z axis, symmetry considerations dictate that **D** is in the radial $\hat{\mathbf{r}}$ direction and cannot depend on ϕ or z. Thus, $\mathbf{D} = \hat{\mathbf{r}} D_r$. Therefore, we construct a finite cylindrical Gaussian surface of radius r and height h, which is concentric around the line of charge (**Fig.2-10**). The total charge contained within the cylinder is $Q = \rho_\ell h$. Since **D** is along $\hat{\mathbf{r}}$, the top and bottom surfaces of the cylinder do not contribute to the surface integral on the left-hand side of Eq. (4.29); that is, only the curved surface contributes to the integral. Hence,

$$\int_{z=0}^{h}\int_{\phi=0}^{2\pi} \hat{\mathbf{r}} D_r \cdot \hat{\mathbf{r}}\, r\, d\phi\, dz = \rho_\ell h$$

or

$$2\pi h D_r r = \rho_\ell h,$$

which yields

$$\mathbf{E} = \frac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}}\,\frac{D_r}{\varepsilon_0} = \hat{\mathbf{r}}\,\frac{\rho_\ell}{2\pi\varepsilon_0 r}. \tag{4.33}$$

(infinite line charge)

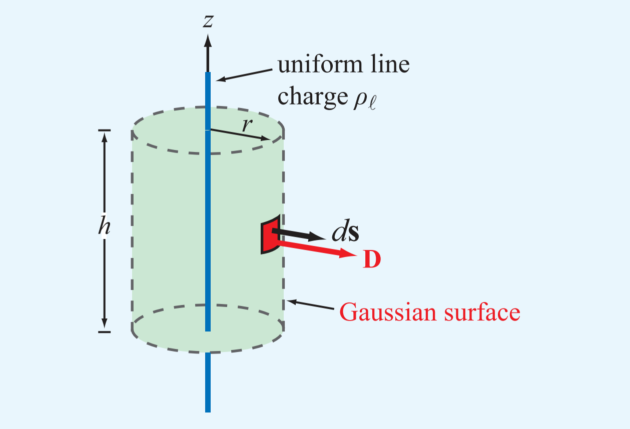

Figure2-10 Gaussian surface around an infinitely long line of charge (Example2-6).

Note that Eq. (2.33) is applicable for any infinite line of charge, regardless of its location and direction, as long as $\hat{\mathbf{r}}$ is properly defined as the radial distance vector from the line charge to the observation point (i.e., $\hat{\mathbf{r}}$ is perpendicular to the line of charge).

Example2-7: Two Infinite Lines of Charge

Figure2-11 depicts the presence of two infinite lines of charge in free space: one residing in the x–y plane parallel to the $\hat{\mathbf{x}}$ axis and carrying charge density $\rho_{\ell_1} = 1$ (nC/m), and a second one residing in the y–z plane parallel to the y axis and carrying a charge density $\rho_{\ell_2} = -2$ (nC/m). Determine the electric field at the origin.

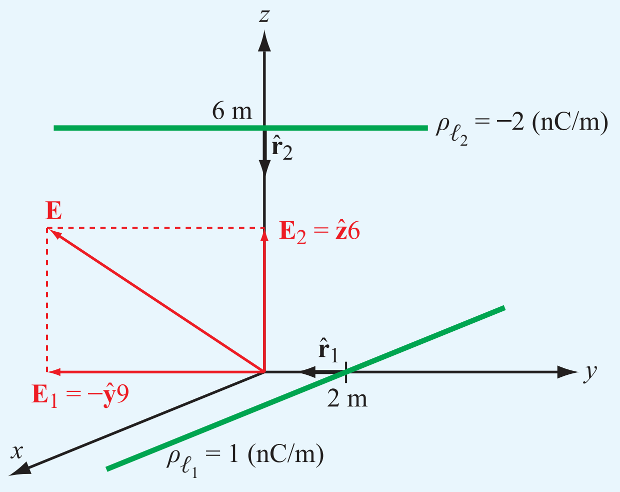

Figure2-11 Two infinite lines of charge (Example 4-7).

Solution: The electric field $\mathbf{E}$ is the sum of two electric field components:

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2,$$

where $\mathbf{E}_1$ and $\mathbf{E}_2$ are the electric fields due to line charges 1 and 2, respectively. According to Eq. (4.33), the direction of the electric field $\hat{\mathbf{r}}$ is perpendicular to the direction of the line charge and points away from the line of charge (if ρ_ℓ is positive). Hence, for the first line of charge, $\rho_{\ell_1} = 1$ (nC/m), $\hat{\mathbf{r}}_1 = -\hat{\mathbf{y}}$, $r_1 = 2$, and

$$\mathbf{E}_1 = \frac{\hat{\mathbf{r}}_1 \rho_{\ell_1}}{2\pi\varepsilon_0 r_1} = \frac{-\hat{\mathbf{y}}\, 10^{-9}}{2\pi \times \frac{1}{36\pi} \times 10^{-9} \times 2} = -\hat{\mathbf{y}}\, 9 \quad \text{V/m}.$$

Similarly, for the second line of charge, $\rho_{\ell_2} = -2$ (nC/m), $\hat{\mathbf{r}}_2 = -\hat{\mathbf{z}}$, $r_2 = 6$, and

$$\mathbf{E}_2 = \frac{\hat{\mathbf{r}}_2 \rho_{\ell_2}}{2\pi\varepsilon_0 r_2} = \frac{-\hat{\mathbf{z}}(-2) \times 10^{-9}}{2\pi \times \frac{1}{36\pi} \times 10^{-9} \times 6} = \hat{\mathbf{z}}\, 6 \quad \text{V/m}.$$

Hence,

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = (-\hat{\mathbf{y}}\, 9 + \hat{\mathbf{z}}\, 6) \quad \text{V/m}.$$

Concept Question2-7: Explain Gauss's law. Under what circumstances is it useful?

Concept Question2-8: How should one choose a Gaussian surface?

Exercise2-7: Two infinite lines, each carrying a uniform charge density ρ_ℓ, reside in free space parallel to the z axis at $x = 1$ and $x = -1$. Determine $\mathbf{E}$ at an arbitrary point along the y axis.

Answer: $\mathbf{E} = \hat{\mathbf{y}}\rho_\ell y / \left[\pi\varepsilon_0(y^2+1)\right]$. (See EM.)

Exercise 4-8: A thin spherical shell of radius a carries a uniform surface charge density ρ_s. Use Gauss's law to determine $\mathbf{E}$ everywhere in free space.

Answer: $\mathbf{E} = \begin{cases} 0 & \text{for } R < a; \\ \hat{\mathbf{R}}\rho_s a^2/(\varepsilon R^2) & \text{for } R > a. \end{cases}$ (See EM.)

Exercise 2-9: A spherical volume of radius a contains a uniform volume charge density ρ_v. Use Gauss's law to determine $\mathbf{D}$ for (a) $R \le a$ and (b) $R \ge a$.

Answer: (a) $\mathbf{D} = \hat{\mathbf{R}}\rho_v R/3$,
(b) $\mathbf{D} = \hat{\mathbf{R}}\rho_v a^3/(3R^2)$. (See EM.)

2-5 Electric Scalar Potential

The operation of an electric circuit usually is described in terms of the currents flowing through its branches and the voltages at its nodes. The voltage difference V between two points in a circuit represents the amount of work, or *potential* energy, required to move a unit charge from one to the other.

▶ The term "voltage" is short for "voltage potential" and synonymous with ***electric potential***. ◀

Even though when analyzing a circuit we may not consider the electric fields present in the circuit, it is in fact the existence of these fields that gives rise to voltage differences across circuit elements such as resistors or capacitors. The relationship between the electric field $\mathbf{E}$ and the electric potential V is the subject of this section.

2-5.1 Electric Potential as a Function of Electric Field

We begin by considering the simple case of a positive charge q in a uniform electric field $\mathbf{E} = -\hat{\mathbf{y}}E$ in the $-y$ direction (Fig.2-12). The presence of the field $\mathbf{E}$ exerts a force $\mathbf{F}_e = q\mathbf{E}$ on the charge in the $-y$ direction. To move the charge along the positive y direction (against the force $\mathbf{F}_e$), we need to provide an external force $\mathbf{F}_{ext}$ to counteract $\mathbf{F}_e$, which requires the expenditure of energy. To move q without acceleration (at constant speed), the net force acting on the charge must be zero, which means that $\mathbf{F}_{ext} + \mathbf{F}_e = 0$, or

$$\mathbf{F}_{ext} = -\mathbf{F}_e = -q\mathbf{E}. \qquad (4.34)$$

The work done (or energy expended) in moving any object a vector differential distance $d\mathbf{l}$ while exerting a force $\mathbf{F}_{ext}$ is

$$dW = \mathbf{F}_{ext} \cdot d\mathbf{l} = -q\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J)}. \qquad (4.35)$$

Work (or energy) is measured in joules (J). If the charge is moved a distance dy along $\hat{\mathbf{y}}$, then

$$dW = -q(-\hat{\mathbf{y}}E) \cdot \hat{\mathbf{y}}\, dy = qE\, dy. \qquad (4.36)$$

The differential electric potential energy dW per unit charge is called the ***differential electric potential*** (or differential voltage) dV. That is,

$$dV = \frac{dW}{q} = -\mathbf{E} \cdot d\mathbf{l} \qquad \text{(J/C or V)}. \qquad (4.37)$$

The unit of V is the volt (V) with 1 V = 1 J/C, and since V is measured in volts, the electric field is expressed in volts per meter (V/m).

The ***potential difference*** corresponding to moving a point charge from point P_1 to point P_2 (Fig.2-13) is obtained by integrating Eq. (4.37) *along any path* between them. That is,

$$\int_{P_1}^{P_2} dV = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \qquad (4.38)$$

or

$$V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}, \qquad (4.39)$$

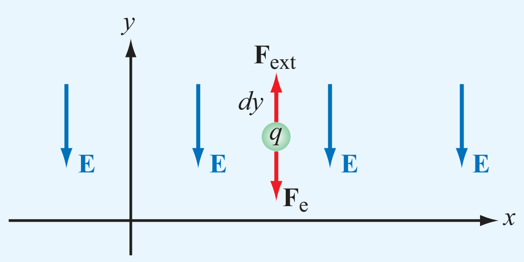

Figure2-12 Work done in moving a charge q a distance dy against the electric field $\mathbf{E}$ is $dW = qE\, dy$.

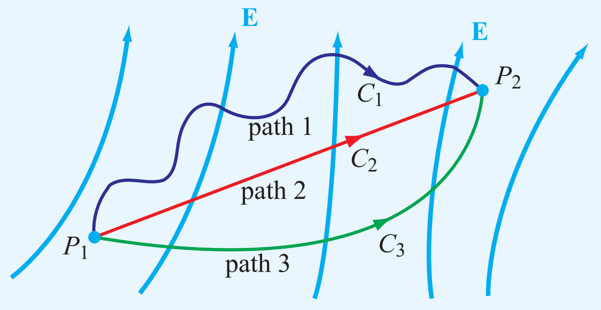

Figure2-13 In electrostatics, the potential difference between P_2 and P_1 is the same irrespective of the path used for calculating the line integral of the electric field between them.

where V_1 and V_2 are the electric potentials at points P_1 and P_2, respectively. The result of the line integral on the right-hand side of Eq. (4.39) is independent of the specific integration path that connects points P_1 and P_2. This follows immediately from the law of conservation of energy. To illustrate with an example, consider a particle in Earth's gravitational field. If the particle is raised from a height h_1 above Earth's surface to height h_2, the particle gains potential energy in an amount proportional to $(h_2 - h_1)$. If instead we were to first raise the particle from height h_1 to a height h_3 greater than h_2, thereby giving it potential energy proportional to $(h_3 - h_1)$, and then let it drop back to height h_2 by expending an energy amount proportional to $(h_3 - h_2)$, its net gain in potential energy would again be proportional to $(h_2 - h_1)$.

The same principle applies to the electric potential energy W and to the potential difference $(V_2 - V_1)$. The voltage difference between two nodes in an electric circuit has the same value regardless of which path in the circuit we follow between the nodes. Moreover, ***Kirchhoff's voltage law*** states that the net voltage drop around a closed loop is zero. If we go from P_1 to P_2 by path 1 in **Fig. 4-13** and then return from P_2 to P_1 by path 2, the right-hand side of Eq. (4.39) becomes a closed contour, and the left-hand side vanishes. In fact, *the line integral of the electrostatic field* **E** *around any closed contour C is zero:*

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0. \qquad \text{(electrostatics)} \qquad (4.40)$$

► A vector field whose line integral along any closed path is zero is called a ***conservative*** or an ***irrotational*** field. Hence, the electrostatic field **E** is conservative. ◄

As we will see later in Chapter 6, if **E** is a time-varying function, it is no longer conservative, and its line integral along a closed path is not necessarily zero.

The conservative property of the electrostatic field can be deduced from Maxwell's second equation, Eq. (4.1b). If $\partial/\partial t = 0$, then

$$\nabla \times \mathbf{E} = 0. \qquad (4.41)$$

If we take the surface integral of $\nabla \times \mathbf{E}$ over an open surface S and then apply Stokes's theorem expressed by Eq. (3.107) to convert the surface integral into a line integral, we obtain

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = \oint_C \mathbf{E} \cdot d\mathbf{l} = 0, \qquad (4.42)$$

where C is a closed contour surrounding S. Thus, Eq. (4.41) is the differential-form equivalent of Eq. (4.40).

We now define what we mean by the electric potential V at a point in space. Before we do so, however, let us revisit our electric-circuit analogue. Just as a node in a circuit cannot be assigned an absolute voltage, a point in space cannot have an absolute electric potential. The voltage of a node in a circuit is measured relative to that of a conveniently chosen reference point to which we have assigned a voltage of zero, which we call *ground*. The same principle applies to the electric potential V. Usually (but not always), the reference point is chosen to be at infinity. That is, in Eq. (4.39) we assume that $V_1 = 0$ when P_1 is at infinity. Therefore, the electric potential V at any point P is

$$V = -\int_\infty^P \mathbf{E} \cdot d\mathbf{l} \qquad \text{(V)}. \qquad (4.43)$$

Example2-8: Computing V from E along Two Paths

A vector field is said to be ***conservative*** if its line integral between two points is the same—irrespective of the path taken between them. In a given region of space, the field **E** is given by

$$\mathbf{E} = \hat{\mathbf{x}}\, x^2 + \hat{\mathbf{y}}\, y^2 + \hat{\mathbf{z}}\, z^2. \qquad (4.44)$$

(a) Confirm that **E** is conservative by demonstrating that $\nabla \times \mathbf{E} = 0$. (b) Compute the potential difference V_{21} between points 1 and 2 in **Fig. 4-14** following the direct path between them. (c) Compute V_{21} by following the path $ABCD$ between points 1 and 2.

Solution: (a) The given electric field has components $E_x = x^2$, $E_y = y^2$, and $E_z = z^2$. Applying the curl operator to **E** gives

$$\nabla \times \mathbf{E} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$
$$= \hat{\mathbf{x}}\left(\frac{\partial z^2}{\partial y} - \frac{\partial y^2}{\partial z}\right) - \hat{\mathbf{y}}\left(\frac{\partial z^2}{\partial x} - \frac{\partial x^2}{\partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial y^2}{\partial x} - \frac{\partial x^2}{\partial y}\right)$$
$$= \hat{\mathbf{x}}(0-0) - \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(0-0) = 0.$$

(b) Voltage V_{21} is given by

$$V_{21} = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}.$$

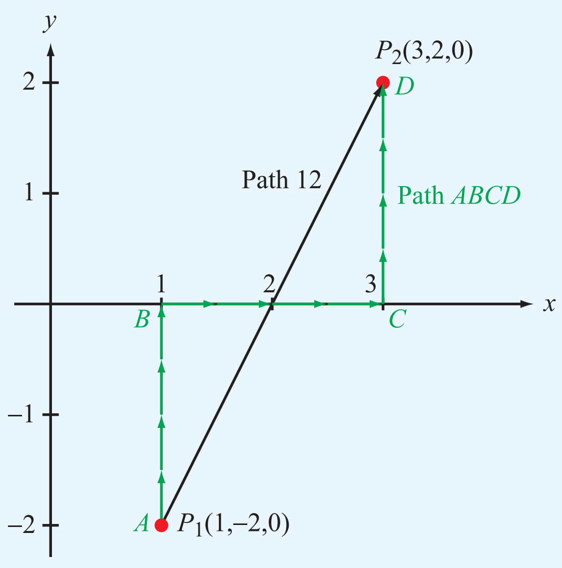

Figure2-14 Computing V_{21} along two paths (Example 4-8).

The straight-line path resides in the x–y plane, so it is described by the linear form $y = ax + b$. At point 1, $x_1 = 1$ and $y_1 = -2$. Hence,

$$-2 = a + b.$$

Similarly, at point 2, $x_2 = 3$ and $y_2 = 2$, so

$$2 = 3a + b.$$

The two equations lead to $a = 2$, $b = -4$, and

$$y = 2x - 4. \tag{4.45}$$

Since path P_1–P_2 is entirely in the x–y plane, we can set $z = 0$ in the expression for $\mathbf{E}$. Also, we can use the relation given by Eq. (4.45) to reduce $\mathbf{E}$ to a single variable:

$$\mathbf{E} = \hat{\mathbf{x}}\, x^2 + \hat{\mathbf{y}}\, y^2 + \hat{\mathbf{z}}\, z^2\big|_{z=0 \text{ and } y=2x-4} = \hat{\mathbf{x}}\, x^2 + \hat{\mathbf{y}}(2x-4)^2. \tag{4.46}$$

In general,

$$d\mathbf{l} = \hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy + \hat{\mathbf{z}}\, dz.$$

In the x–y plane, $dz = 0$, and along the straight-line path given by $y = 2x - 4$,

$$dy = 2\, dx. \tag{4.47}$$

Hence,

$$d\mathbf{l} = \hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, 2\, dx. \tag{4.48}$$

The potential difference is then

$$\begin{aligned} V_{21} &= -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l} \\ &= -\int_{x=1}^{x=3} [\hat{\mathbf{x}}\, x^2 + \hat{\mathbf{y}}(2x-4)^2] \cdot [\hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, 2\, dx] \\ &= -\int_{x=1}^{3} [x^2 + 2(2x-4)^2]\, dx \\ &= -\int_{x=1}^{3} (9x^2 - 32x + 32)\, dx = -14 \qquad \text{(V)}. \end{aligned} \tag{4.49}$$

(c) Path $ABCD$ in **Fig. 4-14** consists of three segments.

A* to *B:

$$\mathbf{E} = \hat{\mathbf{x}}\, x^2 + \hat{\mathbf{y}}\, y^2 + \hat{\mathbf{z}}\, z^2\big|_{x=1,\ z=0} = \hat{\mathbf{x}}\, 1 + \hat{\mathbf{y}}\, y^2, \tag{4.50a}$$

and

$$d\mathbf{l} = \hat{\mathbf{y}}\, dy. \tag{4.50b}$$

B* to *C:

$$\mathbf{E} = \hat{\mathbf{x}}\, x^2 + \hat{\mathbf{y}}\, y^2 + \hat{\mathbf{z}}\, z^2\big|_{y=0,\ z=0} = \hat{\mathbf{x}}\, x^2, \tag{4.51a}$$

and

$$d\mathbf{l} = \hat{\mathbf{x}}\, dx. \tag{4.51b}$$

C* to *D:

$$\mathbf{E} = \hat{\mathbf{x}}\, x^2 + \hat{\mathbf{y}}\, y^2 + \hat{\mathbf{z}}\, z^2\big|_{x=3,\ z=0} = \hat{\mathbf{x}}\, 9 + \hat{\mathbf{y}}\, y^2, \tag{4.52a}$$

and

$$d\mathbf{l} = \hat{\mathbf{y}}\, dy. \tag{4.52b}$$

Hence,

$$\begin{aligned} V_{21} &= -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l} \\ &= -\left[\int_{A@x=1,\ y=-2}^{B@x=1,\ y=0} (\hat{\mathbf{x}}\, 1 + \hat{\mathbf{y}}\, y^2) \cdot \hat{\mathbf{y}}\, dy \right. \\ &\qquad + \int_{B@x=1,\ y=0}^{C@x=3,\ y=0} \hat{\mathbf{x}}\, x^2 \cdot \hat{\mathbf{x}}\, dx \\ &\qquad \left. + \int_{C@x=3,\ y=0}^{D@x=3,\ y=2} (\hat{\mathbf{x}}\, 9 + \hat{\mathbf{y}}\, y^2) \cdot \hat{\mathbf{y}}\, dy \right] \\ &= -\left[\frac{y^3}{3}\bigg|_{-2}^{0} + \frac{x^3}{3}\bigg|_{1}^{3} + \frac{y^3}{3}\bigg|_{0}^{2} \right] \\ &= -\left[+\frac{8}{3} + \frac{27}{3} - \frac{1}{3} + \frac{8}{3} \right] = -14 \qquad \text{(V)}, \end{aligned} \tag{4.53}$$

which is identical with the result given by Eq. (4.49) for the line integral along the straight-line path between points 1 and 2.

2-5.2 Electric Potential Due to Point Charges

The electric field due to a point charge q located at the origin is given by Eq. (4.32) as

$$\mathbf{E} = \hat{\mathbf{R}}\,\frac{q}{4\pi\varepsilon R^2} \qquad \text{(V/m)}. \tag{4.54}$$

The field is radially directed and decays quadratically with the distance R from the observer to the charge.

As was stated earlier, the choice of integration path between the end points in Eq. (4.43) is arbitrary. Hence, we can conveniently choose the path to be along the radial direction $\hat{\mathbf{R}}$, in which case $d\mathbf{l} = \hat{\mathbf{R}}\,dR$ and

$$V = -\int_{\infty}^{R}\left(\hat{\mathbf{R}}\,\frac{q}{4\pi\varepsilon R^2}\right)\cdot\hat{\mathbf{R}}\,dR = \frac{q}{4\pi\varepsilon R} \qquad \text{(V)}. \tag{4.55}$$

If the charge q is at a location other than the origin, say at position vector $\mathbf{R}_1$, then V at observation position vector $\mathbf{R}$ becomes

$$V = \frac{q}{4\pi\varepsilon|\mathbf{R}-\mathbf{R}_1|} \qquad \text{(V)}, \tag{4.56}$$

where $|\mathbf{R}-\mathbf{R}_1|$ is the distance between the observation point and the location of the charge q. The principle of superposition applied previously to the electric field $\mathbf{E}$ also applies to the electric potential V. Hence, for N discrete point charges $q_1, q_2, \ldots, q_N$ residing at position vectors $\mathbf{R}_1, \mathbf{R}_2, \ldots, \mathbf{R}_N$, the electric potential is

$$V = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{N}\frac{q_i}{|\mathbf{R}-\mathbf{R}_i|} \qquad \text{(V)}. \tag{4.57}$$

2-5.3 Electric Potential Due to Continuous Distributions

To obtain expressions for the electric potential V due to continuous charge distributions over a volume $\mathcal{V}'$, across a surface S', or along a line l', we (1) replace q_i in Eq. (4.57) with $\rho_{\rm v}\,d\mathcal{V}'$, $\rho_{\rm s}\,ds'$, and $\rho_\ell\,dl'$, respectively; (2) convert the summation into an integration; and (3) define $R' = |\mathbf{R}-\mathbf{R}_i|$ as the distance between the integration point and the observation point. These steps lead to the following expressions:

$$V = \frac{1}{4\pi\varepsilon}\int_{\mathcal{V}'}\frac{\rho_{\rm v}}{R'}\,d\mathcal{V}' \quad \textbf{(volume distribution)}, \tag{4.58a}$$

$$V = \frac{1}{4\pi\varepsilon}\int_{S'}\frac{\rho_{\rm s}}{R'}\,ds' \quad \textbf{(surface distribution)}, \tag{4.58b}$$

$$V = \frac{1}{4\pi\varepsilon}\int_{l'}\frac{\rho_\ell}{R'}\,dl' \quad \textbf{(line distribution)}. \tag{4.58c}$$

2-5.4 Electric Field as a Function of Electric Potential

In Section 4-5.1, we expressed V in terms of a line integral over $\mathbf{E}$. Now we explore the inverse relationship by reexamining Eq. (4.37):

$$dV = -\mathbf{E}\cdot d\mathbf{l}. \tag{4.59}$$

For a scalar function V, Eq. (3.73) gives

$$dV = \nabla V\cdot d\mathbf{l}, \tag{4.60}$$

where ∇V is the gradient of V. Comparison of Eq. (4.59) with Eq. (4.60) leads to

$$\mathbf{E} = -\nabla V. \tag{4.61}$$

▶ This differential relationship between V and $\mathbf{E}$ allows us to determine $\mathbf{E}$ for any charge distribution by first calculating V and then taking the negative gradient of V to find $\mathbf{E}$. ◀

The expressions for V, given by Eqs. (4.57) to (4.58c), involve scalar sums and scalar integrals, and as such are usually much easier to evaluate than the vector sums and integrals in the expressions for $\mathbf{E}$ derived in Section 4-3 on the basis of Coulomb's law. Thus, even though the electric potential approach for finding $\mathbf{E}$ is a two-step process, it is conceptually and computationally simpler to apply than the direct method based on Coulomb's law.

Example2-9: Electric Field of an Electric Dipole

An ***electric dipole*** consists of two point charges of equal magnitude but opposite polarity separated by a distance d (**Fig. 4-15(a)**). Determine V and $\mathbf{E}$ at any point P given that P is at a distance $R \gg d$ from the dipole center and the dipole resides in free space.

Solution: To simplify the derivation, we align the dipole along the z axis and center it at the origin (**Fig. 4-15(a)**). For the two charges shown in **Fig. 4-15(a)**, application of Eq. (4.57) gives

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{R_1}+\frac{-q}{R_2}\right) = \frac{q}{4\pi\varepsilon_0}\left(\frac{R_2-R_1}{R_1R_2}\right).$$

Since $d \ll R$, the lines labeled R_1 and R_2 in **Fig. 4-15(a)** are approximately parallel to each other, in which case the following approximations apply:

$$R_2 - R_1 \approx d\cos\theta, \qquad R_1R_2 \approx R^2.$$

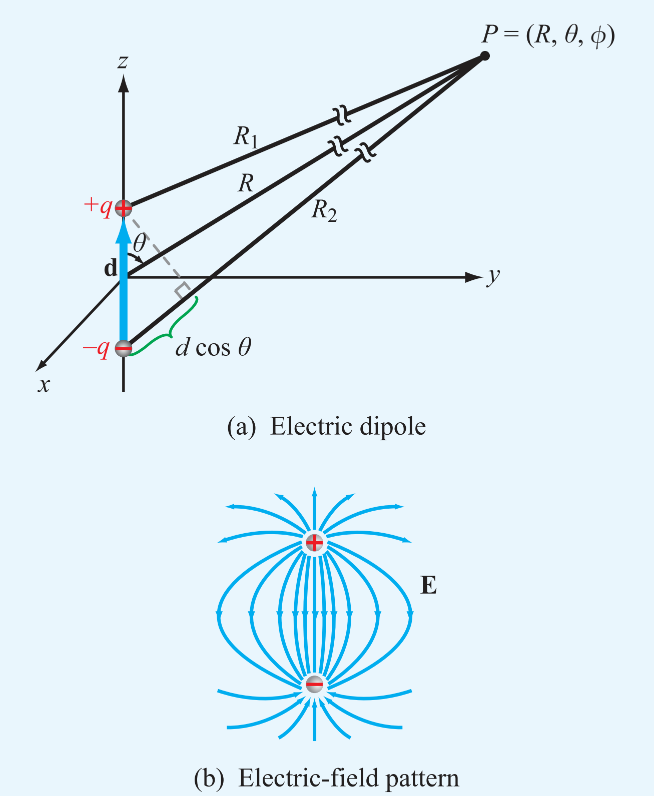

Figure2-15 Electric dipole with dipole moment $\mathbf{p} = q\mathbf{d}$ (Example 4-9).

Hence,

$$V = \frac{qd\cos\theta}{4\pi\varepsilon_0 R^2}. \tag{4.62}$$

To generalize this result to an arbitrarily oriented dipole, note that the numerator of Eq. (4.62) can be expressed as the dot product of $q\mathbf{d}$ (where $\mathbf{d}$ is the distance vector from $-q$ to $+q$) and the unit vector $\hat{\mathbf{R}}$ pointing from the center of the dipole toward the observation point P. That is,

$$qd\cos\theta = q\mathbf{d}\cdot\hat{\mathbf{R}} = \mathbf{p}\cdot\hat{\mathbf{R}}, \tag{4.63}$$

where $\mathbf{p} = q\mathbf{d}$ is called the ***dipole moment***. Using Eq. (4.63) in Eq. (4.62) then gives

$$V = \frac{\mathbf{p}\cdot\hat{\mathbf{R}}}{4\pi\varepsilon_0 R^2} \qquad \textbf{(electric dipole)}. \tag{4.64}$$

In spherical coordinates, Eq. (4.61) is given by

$$\mathbf{E} = -\nabla V = -\left(\hat{\mathbf{R}}\,\frac{\partial V}{\partial R} + \hat{\boldsymbol{\theta}}\,\frac{1}{R}\,\frac{\partial V}{\partial \theta} + \hat{\boldsymbol{\phi}}\,\frac{1}{R\sin\theta}\,\frac{\partial V}{\partial \phi}\right), \tag{4.65}$$

where we have used the expression for ∇V in spherical coordinates given in Appendix C. Upon taking the derivatives of the expression for V given by Eq. (4.62) with respect to R and θ and then substituting the results in Eq. (4.65), we obtain

$$\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 R^3}\,(\hat{\mathbf{R}}\,2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta) \quad \text{(V/m)}. \tag{4.66}$$

We stress that the expressions for V and $\mathbf{E}$ given by Eqs. (4.64) and (4.66) apply only when $R \gg d$. To compute V and $\mathbf{E}$ at points in the vicinity of the two dipole charges, it is necessary to perform all calculations without resorting to the far-distance approximations that led to Eq. (4.62). Such an exact calculation for $\mathbf{E}$ leads to the field pattern shown in **Fig.2-15(b)** .

2-5.5 Poisson's Equation

With $\mathbf{D} = \varepsilon\mathbf{E}$, the differential form of Gauss's law given by Eq. (4.26) may be cast as

$$\nabla\cdot\mathbf{E} = \frac{\rho_{\mathrm{v}}}{\varepsilon}. \tag{4.67}$$

Inserting Eq. (4.61) in Eq. (4.67) gives

$$\nabla\cdot(\nabla V) = -\frac{\rho_{\mathrm{v}}}{\varepsilon}. \tag{4.68}$$

Given Eq. (3.110) for the Laplacian of a scalar function V,

$$\nabla^2 V = \nabla\cdot(\nabla V) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}, \tag{4.69}$$

Eq. (4.68) can be cast in the abbreviated form

$$\nabla^2 V = -\frac{\rho_{\mathrm{v}}}{\varepsilon} \qquad \textbf{(Poisson's equation)}. \tag{4.70}$$

This is known as ***Poisson's equation***. For a volume υ' containing a volume charge density distribution ρ_{v}, the solution for V derived previously and expressed by Eq. (4.58a) as

$$V = \frac{1}{4\pi\varepsilon}\int_{\upsilon'} \frac{\rho_{\mathrm{v}}}{R'}\,d\upsilon' \tag{4.71}$$

satisfies Eq. (4.70). If the medium under consideration contains no charges, Eq. (4.70) reduces to

$$\nabla^2 V = 0 \qquad \textbf{(Laplace's equation)}, \tag{4.72}$$

and it is then referred to as ***Laplace's equation***. Poisson's and Laplace's equations are useful for determining the electrostatic potential V in regions with boundaries on which V is known, such as the region between the plates of a capacitor with a specified voltage difference across it.

Concept Question 4-9: What is a conservative field?

Concept Question 4-10: Why is the electric potential at a point in space always defined relative to the potential at some reference point?

Concept Question 4-11: Explain why Eq. (4.40) is a mathematical statement of Kirchhoff's voltage law.

Concept Question 4-12: Why is it usually easier to compute V for a given charge distribution and then find $\mathbf{E}$ using $\mathbf{E} = -\nabla V$ than to compute $\mathbf{E}$ directly by applying Coulomb's law?

Concept Question 4-13: What is an electric dipole?

Exercise 4-10: Determine the electric potential at the origin due to four 20 μC charges residing in free space at the corners of a 2 m $\times$ 2 m square centered about the origin in the x–y plane.

Answer: $V = \sqrt{2} \times 10^{-5}/(\pi\varepsilon_0)$ (V). (See EM.)

Exercise 4-11: A spherical shell of radius a has a uniform surface charge density ρ_s. Determine (a) the electric potential and (b) the electric field with both at the center of the shell.

Answer: (a) $V = \rho_s a/\varepsilon$ (V), (b) $\mathbf{E} = 0$. (See EM.)

2-6 Conductors

The electromagnetic ***constitutive parameters*** of a material medium are its electrical permittivity ε, magnetic permeability μ, and conductivity σ. A material is said to be ***homogeneous*** if its constitutive parameters do not vary from point to point, and ***isotropic*** if they are independent of direction. Most materials are isotropic, but some crystals are not. Throughout this book, all materials are assumed to be homogeneous and isotropic. This section is concerned with σ, Section 4-7 examines ε, and discussion of μ is deferred to Chapter 5.

► The ***conductivity*** of a material is a measure of how easily electrons can travel through the material under the influence of an externally applied electric field. ◄

Materials are classified as ***conductors*** (metals) or ***dielectrics*** (insulators) according to the magnitudes of their conductivities. A conductor has a large number of loosely attached electrons in the outermost shells of its atoms. In the absence of an external electric field, these free electrons move in random directions and with varying speeds. Their random motion produces zero average current through the conductor. Upon applying an external electric field, however, the electrons migrate from one atom to the next in the direction opposite that of the external field. Their movement gives rise to a ***conduction current density***

$$\mathbf{J} = \sigma\mathbf{E} \quad (\text{A/m}^2) \quad \textbf{(Ohm's law)}, \tag{4.73}$$

where σ is the material's conductivity with units of siemen per meter (S/m).

In yet other materials, called dielectrics, the electrons are tightly bound to the atoms—so much so that it is very difficult to detach them under the influence of an electric field. Consequently, no significant conduction current can flow through them.

► A ***perfect dielectric*** is a material with $\sigma = 0$. In contrast, a ***perfect conductor*** is a material with $\sigma = \infty$. Some materials, called superconductors, exhibit such a behavior. ◄

The conductivity σ of most metals is in the range from 10^6 to 10^7 S/m when compared with 10^{-10} to 10^{-17} S/m for good insulators (**Table 4-1** on p. 194). A class of materials called ***semiconductors*** allow for conduction currents even though their conductivities are much smaller than those of metals. The conductivity of pure germanium, for example, is 2.2 S/m. Tabulated values of σ at room temperature (20 °C) are given in Appendix B for some common materials, and a subset is reproduced in **Table 4-1**.

► The conductivity of a material depends on several factors, including temperature and the presence of impurities. In general, σ of metals increases with decreasing temperature. Most superconductors operate in the neighborhood of absolute zero. ◄

Table 4-1 **Conductivity of some common materials at 20 °C.**

Material	Conductivity, σ (S/m)
Conductors	
Silver	6.2×10^7
Copper	5.8×10^7
Gold	4.1×10^7
Aluminum	3.5×10^7
Iron	10^7
Mercury	10^6
Carbon	3×10^4
Semiconductors	
Pure germanium	2.2
Pure silicon	4.4×10^{-4}
Insulators	
Glass	10^{-12}
Paraffin	10^{-15}
Mica	10^{-15}
Fused quartz	10^{-17}

Concept Question 4-14: What are the electromagnetic constitutive parameters of a material?

Concept Question 4-15: What classifies a material as a conductor, a semiconductor, or a dielectric? What is a superconductor?

Concept Question 4-16: What is the conductivity of a perfect dielectric?

2-6.1 Drift Velocity

The drift velocity $\mathbf{u}_e$ of electrons in a conducting material is related to the externally applied electric field $\mathbf{E}$ through

$$\mathbf{u}_e = -\mu_e \mathbf{E} \qquad \text{(m/s)}, \tag{4.74a}$$

where μ_e is a material property call the ***electron mobility*** with units of (m^2/V·s). In a semiconductor, current flow is due to the movement of both electrons and holes, and since holes are positive-charge carriers, the ***hole drift velocity*** $\mathbf{u}_h$ is in the same direction as $\mathbf{E}$,

$$\mathbf{u}_h = \mu_h \mathbf{E} \qquad \text{(m/s)}, \tag{4.74b}$$

where μ_h is the ***hole mobility***. The mobility accounts for the effective mass of a charged particle and the average distance over which the applied electric field can accelerate it before it is stopped by colliding with an atom and then starts accelerating all over again. From Eq. (4.11), the current density in a medium containing a volume density ρ_v of charges moving with velocity $\mathbf{u}$ is $\mathbf{J} = \rho_v \mathbf{u}$. In the most general case, the current density consists of a component $\mathbf{J}_e$ due to electrons and a component $\mathbf{J}_h$ due to holes. Thus, the total ***conduction current density*** is

$$\mathbf{J} = \mathbf{J}_e + \mathbf{J}_h = \rho_{ve}\mathbf{u}_e + \rho_{vh}\mathbf{u}_h \qquad (\text{A/m}^2), \tag{4.75}$$

where $\rho_{ve} = -N_e e$ and $\rho_{vh} = N_h e$ with N_e and N_h being the number of free electrons and the number of free holes per unit volume and $e = 1.6 \times 10^{-19}$ C is the absolute charge of a single hole or electron. Use of Eqs. (4.74a) and (4.74b) gives

$$\mathbf{J} = (-\rho_{ve}\mu_e + \rho_{vh}\mu_h)\mathbf{E} = \sigma \mathbf{E}, \tag{4.76}$$

where the quantity inside the parentheses is defined as the ***conductivity*** of the material, σ. Thus,

$$\sigma = -\rho_{ve}\mu_e + \rho_{vh}\mu_h = (N_e\mu_e + N_h\mu_h)e \qquad \text{(S/m)}, \tag{4.77a}$$

(semiconductor)

and its unit is siemens per meter (S/m). For a good conductor, $N_h\mu_h \ll N_e\mu_e$, and Eq. (4.77a) reduces to

$$\sigma = -\rho_{ve}\mu_e = N_e\mu_e e \qquad \text{(S/m)}. \tag{4.77b}$$

(good conductor)

► In view of Eq. (4.76), in a ***perfect dielectric*** with $\sigma = 0$, $\mathbf{J} = 0$ regardless of $\mathbf{E}$. Similarly, in a ***perfect conductor*** with $\sigma = \infty$, $\mathbf{E} = \mathbf{J}/\sigma = 0$ regardless of $\mathbf{J}$. ◄

That is,

Perfect dielectric:	$\mathbf{J} = 0$,
Perfect conductor:	$\mathbf{E} = 0$.

Because σ is on the order of 10^6 S/m for most metals, such as silver, copper, gold, and aluminum (Table 4-1), it is common practice to treat them as perfect conductors and to set $\mathbf{E} = 0$ inside them.

A perfect conductor is an ***equipotential*** medium, meaning that the electric potential is the same at every point in the conductor. This property follows from the fact that V_{21}, which is the voltage difference between two points in the conductor, equals the line integral of $\mathbf{E}$ between them, as indicated by Eq. (4.39). Since $\mathbf{E} = 0$ everywhere in the perfect conductor, the voltage difference $V_{21} = 0$. The fact that the conductor is an equipotential medium, however, does not necessarily imply that the potential difference between the conductor and some other conductor is zero. Each conductor is an equipotential medium, but the presence of different distributions of charges on their two surfaces can generate a potential difference between them.

Example2-10: Conduction Current in a Copper Wire

A 2-mm diameter copper wire with conductivity of 5.8×10^7 S/m and electron mobility of 0.0032 (m^2/V·s) is subjected to an electric field of 20 (mV/m). Find (a) the volume charge density of the free electrons, (b) the current density, (c) the current flowing in the wire, (d) the electron drift velocity, and (e) the volume density of the free electrons.

Solution: (a)

$$\rho_{\text{ve}} = -\frac{\sigma}{\mu_{\text{e}}} = -\frac{5.8 \times 10^7}{0.0032} = -1.81 \times 10^{10} \text{ (C/m}^3\text{)}.$$

(b)

$$J = \sigma E = 5.8 \times 10^7 \times 20 \times 10^{-3} = 1.16 \times 10^6 \text{ (A/m}^2\text{)}.$$

(c)

$$\begin{aligned} I &= JA \\ &= J\left(\frac{\pi d^2}{4}\right) = 1.16 \times 10^6 \left(\frac{\pi \times 4 \times 10^{-6}}{4}\right) = 3.64 \text{ A}. \end{aligned}$$

(d)

$$u_{\text{e}} = -\mu_{\text{e}} E = -0.0032 \times 20 \times 10^{-3} = -6.4 \times 10^{-5} \text{ m/s}.$$

The minus sign indicates that $\mathbf{u}_{\text{e}}$ is in the opposite direction of $\mathbf{E}$.

(e)

$$N_{\text{e}} = -\frac{\rho_{\text{ve}}}{e} = \frac{1.81 \times 10^{10}}{1.6 \times 10^{-19}} = 1.13 \times 10^{29} \text{ electrons/m}^3.$$

2-6.2 Resistance

To demonstrate the utility of the point form of Ohm's law, we apply it to derive an expression for the resistance R of a conductor of length l and uniform cross section A, as shown in **Fig. 4-16**. The conductor axis is along the x direction and extends between points x_1 and x_2, with $l = x_2 - x_1$. A voltage V applied across the conductor terminals establishes an electric field $\mathbf{E} = \hat{\mathbf{x}} E_x$; the direction of $\mathbf{E}$ is from the point with higher potential (point 1 in **Fig. 4-16**) to the point with lower potential (point 2). The relation between V and E_x is obtained by applying Eq. (4.39):

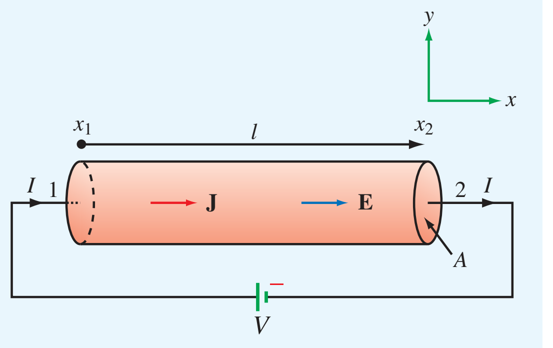

Figure2-16 Linear resistor of cross section A and length l connected to a dc voltage source V.

$$V = V_1 - V_2 = -\int_{x_2}^{x_1} \mathbf{E} \cdot d\mathbf{l} = -\int_{x_2}^{x_1} \hat{\mathbf{x}} E_x \cdot \hat{\mathbf{x}}\, dl = E_x l \quad \text{(V)}. \tag{4.78}$$

Using Eq. (4.73), the current flowing through the cross section A at x_2 is

$$I = \int_A \mathbf{J} \cdot d\mathbf{s} = \int_A \sigma \mathbf{E} \cdot d\mathbf{s} = \sigma E_x A \qquad \text{(A)}. \tag{4.79}$$

From $R = V/I$, the ratio of Eq. (4.78) to Eq. (4.79) gives

$$R = \frac{l}{\sigma A} \qquad (\Omega). \tag{4.80}$$

We now generalize our result for R to any resistor of arbitrary shape by noting that the voltage V across the resistor is equal to the line integral of $\mathbf{E}$ over a path l between two specified points and the current I is equal to the flux of $\mathbf{J}$ through the surface S of the resistor. Thus,

$$R = \frac{V}{I} = \frac{-\displaystyle\int_l \mathbf{E} \cdot d\mathbf{l}}{\displaystyle\int_S \mathbf{J} \cdot d\mathbf{s}} = \frac{-\displaystyle\int_l \mathbf{E} \cdot d\mathbf{l}}{\displaystyle\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}. \tag{4.81}$$

The reciprocal of R is called the ***conductance*** G, and the unit of G is (Ω^{-1}) or siemens (S). For the linear resistor,

$$G = \frac{1}{R} = \frac{\sigma A}{l} \qquad \text{(S)}. \tag{4.82}$$

Example2-11: Conductance of Coaxial Cable

The radii of the inner and outer conductors of a coaxial cable of length l are a and b, respectively (**Fig. 4-17**). The insulation material has conductivity σ. Obtain an expression for G', which is the conductance per unit length of the insulation layer.

Solution: Let I be the total current flowing radially (along $\hat{\mathbf{r}}$) from the inner conductor to the outer conductor through the insulation material. At any radial distance r from the axis of the center conductor, the area through which the current flows is $A = 2\pi rl$. Hence,

$$\mathbf{J} = \hat{\mathbf{r}}\,\frac{I}{A} = \hat{\mathbf{r}}\,\frac{I}{2\pi rl}, \tag{4.83}$$

and from $\mathbf{J} = \sigma\mathbf{E}$,

$$\mathbf{E} = \hat{\mathbf{r}}\,\frac{I}{2\pi\sigma rl}. \tag{4.84}$$

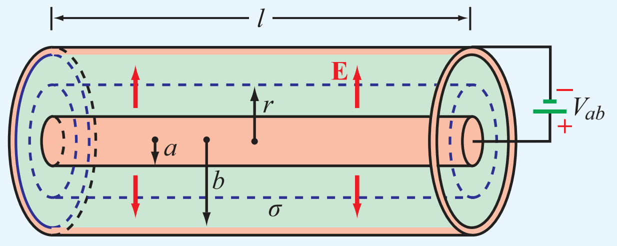

Figure 4-17 Coaxial cable of Example 4-11.

In a resistor, the current flows from higher electric potential to lower potential. Hence, if $\mathbf{J}$ is in the $\hat{\mathbf{r}}$ direction, the inner conductor must be at a potential higher than that at the outer conductor. Accordingly, the voltage difference between the conductors is

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \frac{I}{2\pi\sigma l} \, \frac{\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} \, dr}{r} = \frac{I}{2\pi\sigma l} \ln\left(\frac{b}{a}\right). \tag{4.85}$$

The conductance per unit length is then

$$G' = \frac{G}{l} = \frac{1}{Rl} = \frac{I}{V_{ab} l} = \frac{2\pi\sigma}{\ln(b/a)} \qquad \text{(S/m)}. \tag{4.86}$$

4-6.3 Joule's Law

We now consider the power dissipated in a conducting medium in the presence of an electrostatic field $\mathbf{E}$. The medium contains free electrons and holes with volume charge densities ρ_{ve} and ρ_{vh}, respectively. The electron and hole charge contained in an elemental volume $\Delta\mathcal{V}$ is $q_{\mathrm{e}} = \rho_{\mathrm{ve}} \, \Delta\mathcal{V}$ and $q_{\mathrm{h}} = \rho_{\mathrm{vh}} \, \Delta\mathcal{V}$, respectively. The electric forces acting on q_{e} and q_{h} are $\mathbf{F}_{\mathrm{e}} = q_{\mathrm{e}}\mathbf{E} = \rho_{\mathrm{ve}}\mathbf{E} \, \Delta\mathcal{V}$ and $\mathbf{F}_{\mathrm{h}} = q_{\mathrm{h}}\mathbf{E} = \rho_{\mathrm{vh}}\mathbf{E} \, \Delta\mathcal{V}$. The work (energy) expended by the electric field in moving q_{e} a differential distance Δl_{e} and moving q_{h} a distance Δl_{h} is

$$\Delta W = \mathbf{F}_{\mathrm{e}} \cdot \Delta\mathbf{l}_{\mathrm{e}} + \mathbf{F}_{\mathrm{h}} \cdot \Delta\mathbf{l}_{\mathrm{h}}. \tag{4.87}$$

Power P is measured in watts (W) and is defined as the time rate of change of energy. The power corresponding to ΔW is

$$\begin{aligned} \Delta P = \frac{\Delta W}{\Delta t} &= \mathbf{F}_{\mathrm{e}} \cdot \frac{\Delta\mathbf{l}_{\mathrm{e}}}{\Delta t} + \mathbf{F}_{\mathrm{h}} \cdot \frac{\Delta\mathbf{l}_{\mathrm{h}}}{\Delta t} \\ &= \mathbf{F}_{\mathrm{e}} \cdot \mathbf{u}_{\mathrm{e}} + \mathbf{F}_{\mathrm{h}} \cdot \mathbf{u}_{\mathrm{h}} \\ &= (\rho_{\mathrm{ve}}\mathbf{E} \cdot \mathbf{u}_{\mathrm{e}} + \rho_{\mathrm{vh}}\mathbf{E} \cdot \mathbf{u}_{\mathrm{h}}) \, \Delta\mathcal{V} = \mathbf{E} \cdot \mathbf{J} \, \Delta\mathcal{V}, \end{aligned} \tag{4.88}$$

where $\mathbf{u}_{\mathrm{e}} = \Delta\mathbf{l}_{\mathrm{e}}/\Delta t$ and $\mathbf{u}_{\mathrm{h}} = \Delta\mathbf{l}_{\mathrm{h}}/\Delta t$ are the electron and hole drift velocities, respectively. Equation (4.75) was used in the last step of the derivation leading to Eq. (4.88). For a volume $\mathcal{V}$, the total dissipated power is

$$P = \int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{J} \, d\mathcal{V} \quad \text{(W)} \quad \textbf{(Joule's law)}, \tag{4.89}$$

and in view of Eq. (4.73),

$$P = \int_{\mathcal{V}} \sigma |\mathbf{E}|^2 \, d\mathcal{V} \qquad \text{(W)}. \tag{4.90}$$

Equation (4.89) is a mathematical statement of ***Joule's law***. For the resistor example considered earlier, $|\mathbf{E}| = E_x$ and its volume is $\mathcal{V} = lA$. Separating the volume integral in Eq. (4.90) into a product of a surface integral over A and a line integral over l, we have

$$\begin{aligned} P = \int_{\mathcal{V}} \sigma |\mathbf{E}|^2 \, d\mathcal{V} &= \int_A \sigma E_x \, ds \int_l E_x \, dl \\ &= (\sigma E_x A)(E_x l) = IV \qquad \text{(W)}, \end{aligned} \tag{4.91}$$

where use was made of Eq. (4.78) for the voltage V and Eq. (4.79) for the current I. With $V = IR$, we obtain the familiar expression

$$P = I^2 R \qquad \text{(W)}. \tag{4.92}$$

Concept Question 4-17: What is the fundamental difference between an insulator, a semiconductor, and a conductor?

Concept Question 4-18: Show that the power dissipated in the coaxial cable of **Fig. 4-17** is

$$P = \frac{I^2 \ln(b/a)}{2\pi\sigma l}.$$

Exercise 4-14: A 50-m long copper wire has a circular cross section with radius $r = 2$ cm. Given that the conductivity of copper is 5.8×10^7 S/m, determine (a) the resistance R of the wire and (b) the power dissipated in the wire if the voltage across its length is 1.5 mV.

Answer: (a) $R = 6.9 \times 10^{-4}$ Ω, (b) $P = 3.3$ mW. (See EM.)

Exercise 4-15: Repeat part (b) of Exercise 4-14 by applying Eq. (4.90). (See EM.)

4-7 Dielectrics

The fundamental difference between a conductor and a dielectric is that electrons in the outermost atomic shells of

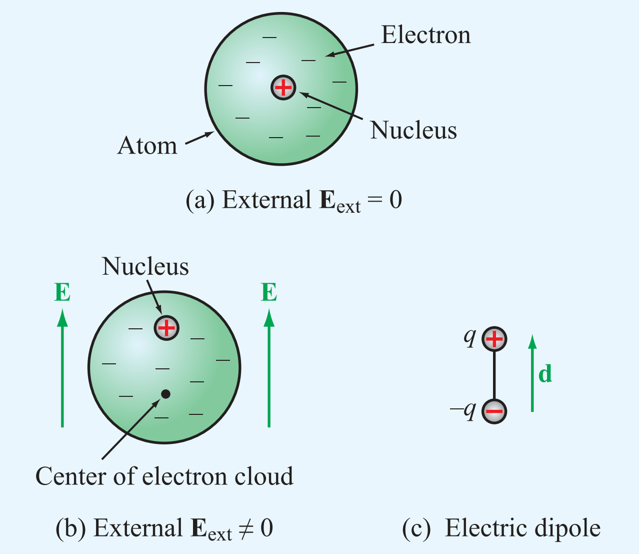

Figure 4-18 In the absence of an external electric field **E**, the center of the electron cloud is co-located with the center of the nucleus, but when a field is applied, the two centers are separated by a distance d.

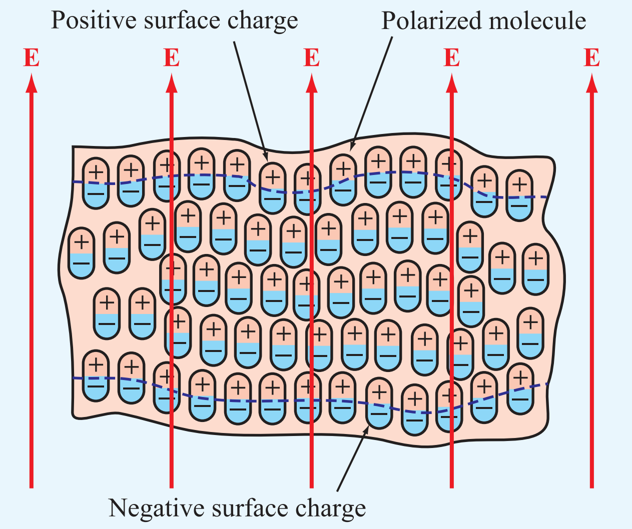

Figure 4-19 A dielectric medium polarized by an external electric field **E**.

a conductor are only weakly tied to atoms and hence can freely migrate through the material, whereas in a dielectric they are strongly bound to the atom. In the absence of an electric field, the electrons in ***nonpolar*** molecules form a symmetrical cloud around the nucleus, with the center of the cloud coinciding with the nucleus (**Fig. 4-18(a)**). The electric field generated by the positively charged nucleus attracts and holds the electron cloud around it, and the mutual repulsion of the electron clouds of adjacent atoms shapes its form. When a conductor is subjected to an externally applied electric field, the most loosely bound electrons in each atom can jump from one atom to the next, thereby setting up an electric current. In a dielectric, however, an externally applied electric field **E** cannot effect mass migration of charges since none are able to move freely. Instead, **E** will ***polarize*** the atoms or molecules in the material by moving the center of the electron cloud away from the nucleus (**Fig.2-18(b)**). The polarized atom or molecule may be represented by an electric dipole consisting of charges $+q$ in the nucleus and $-q$ at the center of the electron cloud (**Fig.2-18(c)**). Each such dipole sets up a small electric field pointing from the positively charged nucleus to the center of the equally but negatively charged electron cloud. This ***induced*** electric field, called a ***polarization*** field, generally is weaker than and opposite in direction to **E**. Consequently, the net electric field present in the dielectric material is smaller than **E**. At the microscopic level, each dipole exhibits a dipole moment similar to that described in Example 4-9. Within a block of dielectric material subject to a uniform external field, the dipoles align themselves linearly, as shown in **Fig. 4-19**. Along the upper and lower edges of the material, the dipole arrangement exhibits positive and negative surface charge densities, respectively.

It is important to stress that this description applies to only nonpolar molecules that do not have permanent dipole moments. Nonpolar molecules become polarized only when an external electric field is applied; when the field is removed, the molecules return to their original unpolarized state.

In polar materials such as water, the molecules possess built-in ***permanent dipole moments*** that are randomly oriented in the absence of an applied electric field, and owing to their random orientations, the dipoles of polar materials produce no net macroscopic dipole moment (at the macroscopic scale, each point in the material represents a small volume containing thousands of molecules). Under the influence of an applied field, the permanent dipoles tend to align themselves along the direction of the electric field in a manner similar to that shown in **Fig.2-19** for nonpolar materials.

2-7.1 Polarization Field

In free space $\mathbf{D} = \varepsilon_0 \mathbf{E}$, the presence of microscopic dipoles in a dielectric material alters that relationship to

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \tag{4.93}$$

where **P**, called the ***electric polarization field***, accounts for the polarization properties of the material. The polarization field is produced by the electric field **E** and depends on the material properties. A dielectric medium is said to be ***linear*** if the magnitude of the induced polarization field **P** is directly proportional to the magnitude of **E**, and ***isotropic*** if **P** and **E** are in the same direction. Some crystals allow more polarization to take place along certain directions, such as the crystal axes, than along others. In such ***anisotropic*** dielectrics, **E** and **P** may have different directions. A medium is said to be ***homogeneous*** if its constitutive parameters (ε, μ, and σ) are constant throughout the medium. Our present treatment will be limited to media that are linear, isotropic, and homogeneous. For such media, **P** is directly proportional to **E** and is expressed as

$$\mathbf{P} = \varepsilon_0 \chi_e \mathbf{E}, \tag{4.94}$$

where χ_e is called the ***electric susceptibility*** of the material. Inserting Eq. (4.94) into Eq. (4.93), we have

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi_e \mathbf{E} = \varepsilon_0 (1 + \chi_e)\mathbf{E} = \varepsilon \mathbf{E}, \tag{4.95}$$

which defines the permittivity ε of the material as

$$\varepsilon = \varepsilon_0 (1 + \chi_e). \tag{4.96}$$

It is often convenient to characterize the permittivity of a material relative to that of free space, ε_0; this is accommodated by the relative permittivity $\varepsilon_r = \varepsilon / \varepsilon_0$. Values of ε_r are listed in **Table 4-2** for a few common materials, and a longer list is given in Appendix B. In free space $\varepsilon_r = 1$, and *for most conductors*, $\varepsilon_r \approx 1$. The dielectric constant of air is approximately 1.0006 at sea level and decreases toward unity with increasing altitude. Except in some special circumstances, such as when calculating electromagnetic wave refraction (bending) through the atmosphere over long distances, *air can be treated as if it were free space.*

2-7.2 Dielectric Breakdown

The preceding dielectric–polarization model presumes that the magnitude of **E** does not exceed a certain critical value, which is known as the ***dielectric strength*** E_{ds} of the material. Beyond this, electrons will detach from the molecules and accelerate through the material in the form of a conduction current. When this happens, sparking can occur, and the dielectric material can sustain permanent damage due to electron collisions with the molecular structure. This abrupt change in behavior is called ***dielectric breakdown***.

▶ The dielectric strength E_{ds} is the largest magnitude of **E** that the material can sustain without breakdown. ◀

Dielectric breakdown can occur in gases, liquids, and solids. The dielectric strength E_{ds} depends on the material composition, as well as other factors such as temperature and humidity. For air, E_{ds} is roughly 3 (MV/m); for glass, 25 to 40 (MV/m); and for mica, 200 (MV/m) (see **Table 4-2**).

A charged thundercloud at electric potential V relative to the ground induces an electric field $E = V/d$ in the air beneath it, where d is the height of the cloud base above the ground. If V is sufficiently large so that E exceeds the dielectric strength of air, ionization occurs and a lightning discharge follows. The ***breakdown voltage*** V_{br} of a parallel-plate capacitor is discussed in Example 4-12.

Example2-12: Dielectric Breakdown

In a parallel-plate capacitor with a separation d between the conducting plates, the electric field E in the dielectric material

Table2-2 **Relative permittivity (dielectric constant) and dielectric strength of common materials.**

Material	Relative Permittivity, ε_r	Dielectric Strength, E_{ds} (MV/m)
Air (at sea level)	1.0006	3
Petroleum oil	2.1	12
Polystyrene	2.6	20
Glass	4.5–10	25–40
Quartz	3.8–5	30
Bakelite	5	20
Mica	5.4–6	200

Note: $\varepsilon = \varepsilon_r \varepsilon_0$ and $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.

separating the two plates is related to the voltage V between the two plates by

$$E = \frac{V}{d}.$$

The breakdown voltage $V_{\rm br}$ corresponds to the value of V at which $E = E_{\rm ds}$, where $E_{\rm ds}$ is the dielectric strength of the material contained between the plates. That is,

$$V_{\rm br} = E_{\rm ds} d.$$

If V exceeds $V_{\rm br}$, the electric charges will "spark" their way between the two plates.

A thin capacitor filled with quartz operates at 60 V. If $d = 0.01$ mm, what is the breakdown voltage, and how does it compare with the operating voltage?

Solution: From **Table2-2** , $E_{\rm ds} = 30 \times 10^6$ V/m for quartz. Hence, the breakdown voltage is

$$V_{\rm br} = E_{\rm ds} d = 30 \times 10^6 \times 10^{-5} = 300 \text{ V},$$

which is much higher than the operating voltage of 60 V. Therefore, the capacitor should experience no issues with dielectric breakdown.

Concept Question 4-19: What is a polar material? A nonpolar material?

Concept Question 4-20: Do **D** and **E** always point in the same direction? If not, when do they not?

Concept Question 4-21: What happens when dielectric breakdown occurs?

2-8 Electric Boundary Conditions

► A vector field is said to be ***spatially continuous*** if it does not exhibit abrupt changes in either magnitude or direction as a function of position. ◄

Even though the electric field may be continuous in adjoining dissimilar media, it may well be discontinuous at the boundary between them. Boundary conditions specify how the components of fields tangential and normal to an interface between two media relate across the interface Here we derive a general set of boundary conditions for **E**, **D**, and **J** that is applicable at the interface between *any* two dissimilar media—be they two dielectrics or a conductor and a dielectric. Of course, any of the dielectrics may be free space. Even though these boundary conditions are derived assuming electrostatic conditions, they remain valid for time-varying electric fields as well. **Figure2-20** shows an interface between medium 1 with permittivity ε_1 and medium 2 with permittivity ε_2. In the general case, the interface may contain a surface charge density $\rho_{\rm s}$ (unrelated to the dielectric polarization charge density).

To derive the boundary conditions for the tangential components of **E** and **D**, we consider the closed rectangular loop *abcda* shown in **Fig.2-20** and apply the conservative property of the electric field expressed by Eq. (4.40), which states that the line integral of the electrostatic field around a closed path is always zero. By letting $\Delta h \to 0$, the contributions to the line integral by segments bc and da vanish. Hence,

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = \int_a^b \mathbf{E}_1 \cdot \hat{\boldsymbol{\ell}}_1 \, dl + \int_c^d \mathbf{E}_2 \cdot \hat{\boldsymbol{\ell}}_2 \, dl = 0, \qquad (4.97)$$

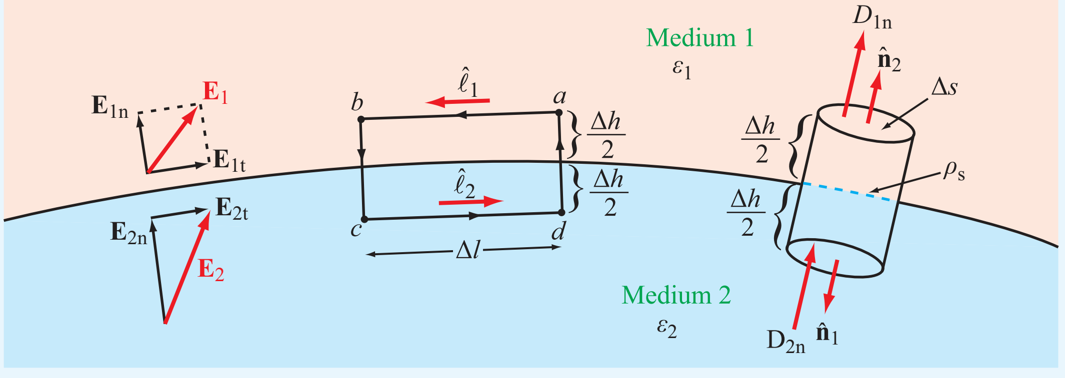

Figure2-20 Interface between two dielectric media.

where $\hat{\boldsymbol{\ell}}_1$ and $\hat{\boldsymbol{\ell}}_2$ are unit vectors along segments *ab* and *cd* and $\mathbf{E}_1$ and $\mathbf{E}_2$ are the electric fields in media 1 and 2. Next, we decompose $\mathbf{E}_1$ and $\mathbf{E}_2$ into components tangential and normal to the boundary (**Fig.2-20**),

$$\mathbf{E}_1 = \mathbf{E}_{1\mathrm{t}} + \mathbf{E}_{1\mathrm{n}}, \tag{4.98a}$$

$$\mathbf{E}_2 = \mathbf{E}_{2\mathrm{t}} + \mathbf{E}_{2\mathrm{n}}. \tag{4.98b}$$

Noting that $\hat{\boldsymbol{\ell}}_1 = -\hat{\boldsymbol{\ell}}_2$, it follows that

$$(\mathbf{E}_1 - \mathbf{E}_2) \cdot \hat{\boldsymbol{\ell}}_1 = 0. \tag{4.99}$$

In other words, the component of $\mathbf{E}_1$ along $\hat{\boldsymbol{\ell}}_1$ equals that of $\mathbf{E}_2$ along $\hat{\boldsymbol{\ell}}_1$, for all $\hat{\boldsymbol{\ell}}_1$ tangential to the boundary. Hence,

$$\mathbf{E}_{1\mathrm{t}} = \mathbf{E}_{2\mathrm{t}} \qquad \text{(V/m)}. \tag{4.100}$$

► Thus, the tangential component of the electric field is **continuous** across the boundary between any two media. ◄

Upon decomposing $\mathbf{D}_1$ and $\mathbf{D}_2$ into tangential and normal components (in the manner of Eq. (4.98)) and noting that $\mathbf{D}_{1\mathrm{t}} = \varepsilon_1 \mathbf{E}_{1\mathrm{t}}$ and $\mathbf{D}_{2\mathrm{t}} = \varepsilon_2 \mathbf{E}_{2\mathrm{t}}$, the boundary condition on the tangential component of the electric flux density is

$$\frac{\mathbf{D}_{1\mathrm{t}}}{\varepsilon_1} = \frac{\mathbf{D}_{2\mathrm{t}}}{\varepsilon_2}. \tag{4.101}$$

Next, we apply Gauss's law, as expressed by Eq. (4.29), to determine boundary conditions on the normal components of $\mathbf{E}$ and $\mathbf{D}$. According to Gauss's law, the total outward flux of $\mathbf{D}$ through the three surfaces of the small cylinder shown in **Fig.2-20** must equal the total charge enclosed in the cylinder. By letting the cylinder's height $\Delta h \to 0$, the contribution to the total flux through the side surface goes to zero. Also, even if each of the two media happens to contain free charge densities, the only charge remaining in the collapsed cylinder is that distributed on the boundary. Thus, $Q = \rho_\mathrm{s}\, \Delta s$, and

$$\begin{aligned}\oint_S \mathbf{D} \cdot d\mathbf{s} &= \int_{\text{top}} \mathbf{D}_1 \cdot \hat{\mathbf{n}}_2 \, ds + \int_{\text{bottom}} \mathbf{D}_2 \cdot \hat{\mathbf{n}}_1 \, ds \\ &= \rho_\mathrm{s}\, \Delta s, \end{aligned} \tag{4.102}$$

where $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are the outward normal unit vectors of the bottom and top surfaces, respectively. It is important to remember that ***the normal unit vector at the surface of any medium is always defined to be in the outward direction away from that medium.*** Since $\hat{\mathbf{n}}_1 = -\hat{\mathbf{n}}_2$, Eq. (4.102) simplifies to

$$\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_\mathrm{s} \qquad (\text{C/m}^2). \tag{4.103}$$

If $D_{1\mathrm{n}}$ and $D_{2\mathrm{n}}$ denote as the normal components of $\mathbf{D}_1$ and $\mathbf{D}_2$ along $\hat{\mathbf{n}}_2$, we have

$$D_{1\mathrm{n}} - D_{2\mathrm{n}} = \rho_\mathrm{s} \qquad (\text{C/m}^2). \tag{4.104}$$

► The normal component of $\mathbf{D}$ changes abruptly at a charged boundary between two different media in an amount equal to the surface charge density. If no charge exists at the boundary, then D_n is continuous across the boundary. ◄

The corresponding boundary condition for $\mathbf{E}$ is

$$\hat{\mathbf{n}}_2 \cdot (\varepsilon_1 \mathbf{E}_1 - \varepsilon_2 \mathbf{E}_2) = \rho_\mathrm{s}, \tag{4.105a}$$

or equivalently

$$\varepsilon_1 E_{1\mathrm{n}} - \varepsilon_2 E_{2\mathrm{n}} = \rho_\mathrm{s}. \tag{4.105b}$$

In summary, (1) the conservative property of $\mathbf{E}$,

$$\nabla \times \mathbf{E} = 0 \quad \longleftrightarrow \quad \oint_C \mathbf{E} \cdot d\mathbf{l} = 0, \tag{4.106}$$

led to the result that $\mathbf{E}$ has a continuous tangential component across a boundary, and (2) the divergence property of $\mathbf{D}$,

$$\nabla \cdot \mathbf{D} = \rho_\mathrm{v} \quad \longleftrightarrow \quad \oint_S \mathbf{D} \cdot d\mathbf{s} = Q, \tag{4.107}$$

led to the result that the normal component of $\mathbf{D}$ changes by ρ_s across the boundary. A summary of the conditions that apply at the boundary between different types of media is given in **Table 4-3**.

Table2-3 **Boundary conditions for the electric fields.**

Field Component	Any Two Media	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential E	$\mathbf{E}_{1\mathrm{t}} = \mathbf{E}_{2\mathrm{t}}$	$\mathbf{E}_{1\mathrm{t}} = \mathbf{E}_{2\mathrm{t}} = 0$	
Tangential D	$\mathbf{D}_{1\mathrm{t}}/\varepsilon_1 = \mathbf{D}_{2\mathrm{t}}/\varepsilon_2$	$\mathbf{D}_{1\mathrm{t}} = \mathbf{D}_{2\mathrm{t}} = 0$	
Normal E	$\varepsilon_1 E_{1\mathrm{n}} - \varepsilon_2 E_{2\mathrm{n}} = \rho_\mathrm{s}$	$E_{1\mathrm{n}} = \rho_\mathrm{s}/\varepsilon_1$	$E_{2\mathrm{n}} = 0$
Normal D	$D_{1\mathrm{n}} - D_{2\mathrm{n}} = \rho_\mathrm{s}$	$D_{1\mathrm{n}} = \rho_\mathrm{s}$	$D_{2\mathrm{n}} = 0$

Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, which is the outward normal unit vector of medium 2.

Example2-13: Application of Boundary Conditions

The x–y plane is a charge-free boundary separating two dielectric media with permittivities ε_1 and ε_2, as shown in Fig.2-21 . If the electric field in medium 1 is

$$E_{1x} + \hat{\mathbf{y}}E_{1y} + \hat{\mathbf{z}}E_{1z},$$

and $\mathbf{E}_1 = \hat{\mathbf{x}}$, find (a) the electric field $\mathbf{E}_2$ in medium 2 and (b) the angles θ_1 and θ_2.

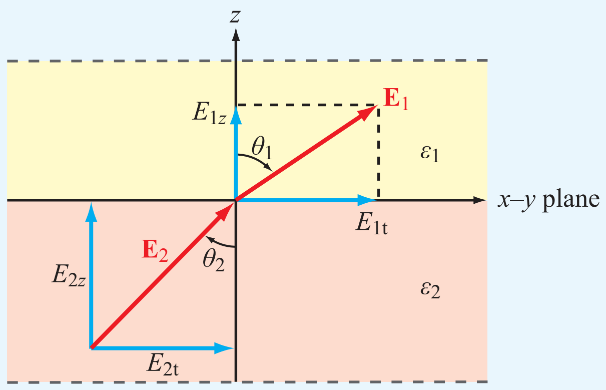

Figure 4-21 Application of boundary conditions at the interface between two dielectric media (Example 4-13).

Solution: (a) Let $\mathbf{E}_2 = \hat{\mathbf{x}}E_{2x} + \hat{\mathbf{y}}E_{2y} + \hat{\mathbf{z}}E_{2z}$. Our task is to find the components of $\mathbf{E}_2$ in terms of the given components of $\mathbf{E}_1$. The normal to the boundary is $\hat{\mathbf{z}}$. Hence, the x and y components of the fields are tangential to the boundary and the z components are normal to the boundary. At a charge-free interface, the tangential components of $\mathbf{E}$ and the normal components of $\mathbf{D}$ are continuous. Consequently,

$$E_{2x} = E_{1x}, \qquad E_{2y} = E_{1y},$$

and

$$D_{2z} = D_{1z} \qquad \text{or} \qquad \varepsilon_2 E_{2z} = \varepsilon_1 E_{1z}.$$

Hence,

$$\mathbf{E}_2 = \hat{\mathbf{x}}E_{1x} + \hat{\mathbf{y}}E_{1y} + \hat{\mathbf{z}}\,\frac{\varepsilon_1}{\varepsilon_2}E_{1z}. \tag{4.108}$$

(b) The tangential components of $\mathbf{E}_1$ and $\mathbf{E}_2$ are

$$E_{1\mathrm{t}} = \sqrt{E_{1x}^2 + E_{1y}^2} \qquad \text{and} \qquad E_{2\mathrm{t}} = \sqrt{E_{2x}^2 + E_{2y}^2}.$$

The angles θ_1 and θ_2 are then given by

$$\tan\theta_1 = \frac{E_{1\mathrm{t}}}{E_{1z}} = \frac{\sqrt{E_{1x}^2 + E_{1y}^2}}{E_{1z}},$$

$$\tan\theta_2 = \frac{E_{2\mathrm{t}}}{E_{2z}} = \frac{\sqrt{E_{2x}^2 + E_{2y}^2}}{E_{2z}} = \frac{\sqrt{E_{1x}^2 + E_{1y}^2}}{(\varepsilon_1/\varepsilon_2)E_{1z}},$$

and the two angles are related by

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{\varepsilon_2}{\varepsilon_1}. \tag{4.109}$$

Exercise2-16: Find $\mathbf{E}_1$ in **Fig. 4-21** if

$$\mathbf{E}_2 = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}3 \text{ (V/m)},$$
$$\varepsilon_1 = 2\varepsilon_0,$$
$$\varepsilon_2 = 8\varepsilon_0,$$

and the boundary is charge-free.

Answer: $\mathbf{E}_1 = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}12$ (V/m). (See EM.)

Exercise2-17: Repeat Exercise 4.16 for a boundary with surface charge density $\rho_s = 3.54 \times 10^{-11}$ (C/m^2).

Answer: $\mathbf{E}_1 = \hat{\mathbf{x}}2 - \hat{\mathbf{y}}3 + \hat{\mathbf{z}}14$ (V/m). (See EM.)

2-8.1 Dielectric-Conductor Boundary

Consider the case when medium 1 is a dielectric and medium 2 is a perfect conductor. In a perfect conductor, because electric fields and fluxes vanish, it follows that $\mathbf{E}_2 = \mathbf{D}_2 = 0$, which implies that components of $\mathbf{E}_2$ and $\mathbf{D}_2$ tangential and normal to the interface are zero. Consequently, from Eq. (4.100) and Eq. (4.104), the fields in the dielectric medium at the boundary with the conductor satisfy

$$E_{1t} = D_{1t} = 0, \qquad (4.110a)$$
$$D_{1n} = \varepsilon_1 E_{1n} = \rho_s. \qquad (4.110b)$$

These two boundary conditions can be combined into

$$\mathbf{D}_1 = \varepsilon_1 \mathbf{E}_1 = \hat{\mathbf{n}}\rho_s, \qquad (4.111)$$

(at conductor surface)

where $\hat{\mathbf{n}}$ is a unit vector directed normally outward from the conducting surface.

▶ The electric field lines point directly away from the conductor surface when ρ_s is positive and directly toward the conductor surface when ρ_s is negative. ◀

Figure 4-22 shows an infinitely long conducting slab placed in a uniform electric field $\mathbf{E}_1$. The media above and below the slab have permittivity ε_1. Because $\mathbf{E}_1$ points away from the upper surface, it induces a positive charge density $\rho_s = \varepsilon_1|\mathbf{E}_1|$ on the upper slab surface. On the bottom surface, $\mathbf{E}_1$ points toward the surface; therefore, the induced charge density is $-\rho_s$. The presence of these surface charges induces an electric field $\mathbf{E}_i$ in the conductor, resulting in a total field $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_i$. To satisfy the condition that $\mathbf{E}$ must be everywhere zero in the conductor, $\mathbf{E}_i$ must equal $-\mathbf{E}_1$.

If we place a metallic sphere in an electrostatic field (**Fig. 4-23**), positive and negative charges accumulate on the upper and lower hemispheres, respectively. The presence of the sphere causes the field lines to bend to satisfy the condition expressed by Eq. (4.111); that is, ***E is always normal to a conductor boundary.***

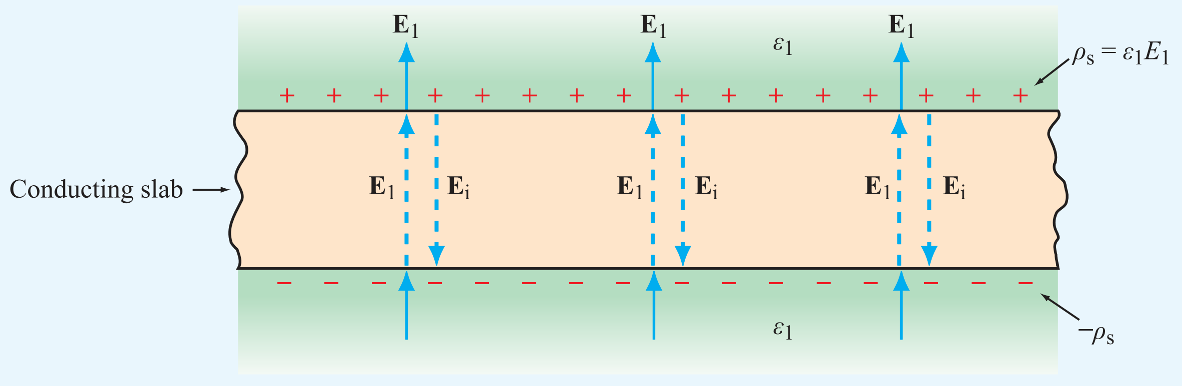

Figure2-22 When a conducting slab is placed in an external electric field $\mathbf{E}_1$, charges that accumulate on the conductor surfaces induce an internal electric field $\mathbf{E}_i = -\mathbf{E}_1$. Consequently, the total field inside the conductor is zero.

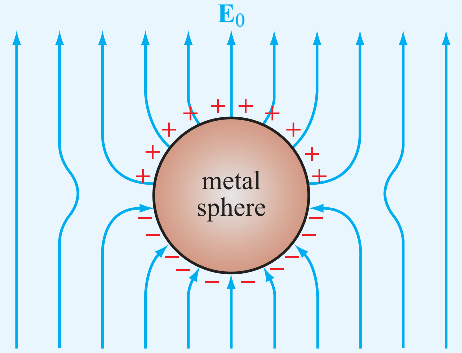

Figure 4-23 Metal sphere placed in an external electric field $\mathbf{E}_0$.

2-8.2 Conductor–Conductor Boundary

We now examine the general case of the boundary between two media—neither of which is a perfect dielectric or a perfect conductor (**Fig.2-24**). Medium 1 has permittivity ε_1 and conductivity σ_1, medium 2 has ε_2 and σ_2, and the interface between them holds a surface charge density ρ_s. For the electric fields, Eqs. (4.100) and (4.105b) give

$$\mathbf{E}_{1\mathrm{t}} = \mathbf{E}_{2\mathrm{t}}, \qquad \varepsilon_1 E_{1\mathrm{n}} - \varepsilon_2 E_{2\mathrm{n}} = \rho_s. \tag{4.112}$$

Since we are dealing with conducting media, the electric fields give rise to current densities $\mathbf{J}_1 = \sigma_1 \mathbf{E}_1$ and $\mathbf{J}_2 = \sigma_2 \mathbf{E}_2$. Hence,

$$\frac{\mathbf{J}_{1\mathrm{t}}}{\sigma_1} = \frac{\mathbf{J}_{2\mathrm{t}}}{\sigma_2}, \qquad \varepsilon_1 \frac{J_{1\mathrm{n}}}{\sigma_1} - \varepsilon_2 \frac{J_{2\mathrm{n}}}{\sigma_2} = \rho_s. \tag{4.113}$$

The tangential current components $\mathbf{J}_{1\mathrm{t}}$ and $\mathbf{J}_{2\mathrm{t}}$ represent currents flowing in the two media in a direction parallel to the

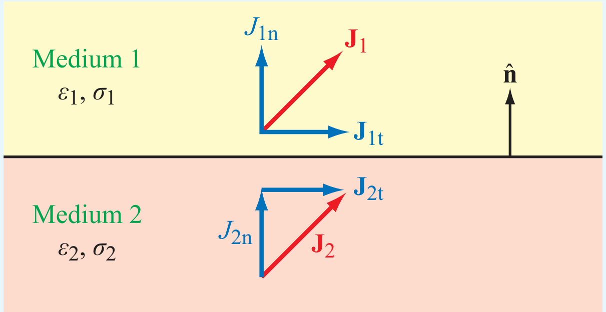

Figure2-24 Boundary between two conducting media.

boundary; hence, there is no transfer of charge between them. This is not the case for the normal components. If $J_{1n} \neq J_{2n}$, then a different amount of charge arrives at the boundary than leaves it. Hence, ρ_s cannot remain constant in time, which violates the condition of electrostatics requiring all fields and charges to remain constant. Consequently, *the normal component of* $\mathbf{J}$ *has to be continuous across the boundary between two different media under electrostatic conditions.* Upon setting $J_{1n} = J_{2n}$ in Eq. (4.113), we have

$$J_{1n}\left(\frac{\varepsilon_1}{\sigma_1} - \frac{\varepsilon_2}{\sigma_2}\right) = \rho_s \quad \textbf{(electrostatics)}. \tag{4.114}$$

Concept Question 2-22: What are the boundary conditions for the electric field at a conductor–dielectric boundary?

Concept Question 2-23:
Under electrostatic conditions, we require $J_{1n} = J_{2n}$ at the boundary between two conductors. Why?

2-9 Capacitance

When separated by an insulating (dielectric) medium, any two conducting bodies, regardless of their shapes and sizes, form a ***capacitor***. If a dc voltage source is connected across them (**Fig. 4-25**) the surfaces of the conductors connected to the positive and negative source terminals accumulate charges $+Q$ and $-Q$, respectively.

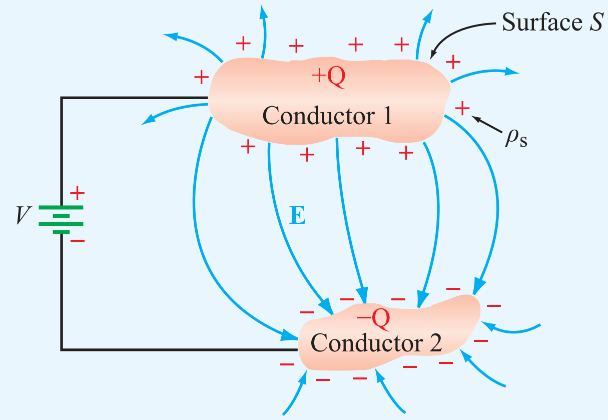

Figure2-25 A dc voltage source connected to a capacitor composed of two conducting bodies.

▶ When a conductor has excess charge, it distributes the charge on its surface in such a manner as to maintain a zero electric field everywhere within the conductor, thereby ensuring that the electric potential is the same at every point in the conductor. ◀

The ***capacitance*** of a two-conductor configuration is defined as

$$C = \frac{Q}{V} \qquad \text{(C/V or F)}, \tag{4.115}$$

where V is the potential (voltage) difference between the conductors. Capacitance is measured in farads (F), which is equivalent to coulombs per volt (C/V).

The presence of free charges on the conductors' surfaces gives rise to an electric field $\mathbf{E}$ (**Fig.2-25**) with field lines originating on the positive charges and terminating on the negative ones. Since the tangential component of $\mathbf{E}$ always vanishes at a conductor's surface, $\mathbf{E}$ is always perpendicular to the conducting surfaces. The normal component of $\mathbf{E}$ at any point on the surface of either conductor is given by

$$E_n = \hat{\mathbf{n}} \cdot \mathbf{E} = \frac{\rho_s}{\varepsilon}, \tag{4.116}$$

(at conductor surface)

where ρ_s is the surface charge density at that point, $\hat{\mathbf{n}}$ is the outward normal unit vector at the same location, and ε is the permittivity of the dielectric medium separating the conductors. The charge Q is equal to the integral of ρ_s over surface S (**Fig. 4-25**):

$$Q = \int_S \rho_s \, ds = \int_S \varepsilon \hat{\mathbf{n}} \cdot \mathbf{E} \, ds = \int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}, \tag{4.117}$$

where use was made of Eq. (4.116). The voltage V is related to $\mathbf{E}$ by Eq. (4.39):

$$V = V_{12} = -\int_{P_2}^{P_1} \mathbf{E} \cdot d\mathbf{l}, \tag{4.118}$$

where points P_1 and P_2 are any two arbitrary points on conductors 1 and 2, respectively. Substituting Eqs. (4.117) and (4.118) into Eq. (4.115) gives

$$C = \frac{\displaystyle\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\displaystyle\int_l \mathbf{E} \cdot d\mathbf{l}} \qquad \text{(F)}, \tag{4.119}$$

where l is the integration path from conductor 2 to conductor 1. To avoid making sign errors when applying Eq. (4.119), it is important to remember that surface S is the $+Q$ surface and P_1 is on S. [Alternatively, if you compute C and it comes out negative, just change its sign.] Because $\mathbf{E}$ appears in both the numerator and denominator of Eq. (4.119), *the value of C obtained for any specific capacitor configuration is always independent of $\mathbf{E}$'s magnitude.* In fact, C depends only on the capacitor geometry (sizes, shapes and relative positions of the two conductors) and the permittivity of the insulating material.

If the material between the conductors is not a perfect dielectric (i.e., if it has a small conductivity σ), then current can flow through the material between the conductors, and the material exhibits a resistance R. The general expression for R for a resistor of arbitrary shape is given by Eq. (4.81):

$$R = \frac{-\displaystyle\int_l \mathbf{E} \cdot d\mathbf{l}}{\displaystyle\int_S \sigma \mathbf{E} \cdot d\mathbf{s}} \qquad (\Omega). \tag{4.120}$$

For a medium with uniform σ and ε, the product of Eqs. (4.119) and (4.120) gives

$$RC = \frac{\varepsilon}{\sigma}. \tag{4.121}$$

This simple relation allows us to find R if C is known, and vice versa.

Example2-14: Capacitance of Parallel-Plate Capacitor

Obtain an expression for the capacitance C of a parallel-plate capacitor comprised of two parallel plates each of surface area A and separated by a distance d. The capacitor is filled with a dielectric material with permittivity ε.

Solution: In **Fig.2-26** , we place the lower plate of the capacitor in the x–y plane and the upper plate in the plane $z = d$. Because of the applied voltage difference V, charges $+Q$ and $-Q$ accumulate on the top and bottom capacitor plates. If the plate dimensions are much larger than the separation d, then these charges distribute themselves quasi-uniformly across the plates, giving rise to a quasi-uniform field between them pointing in the $-\hat{\mathbf{z}}$ direction. In addition, a ***fringing field*** will exist near the capacitor edges, but its effects may be ignored because the bulk of the electric field exists between the plates.

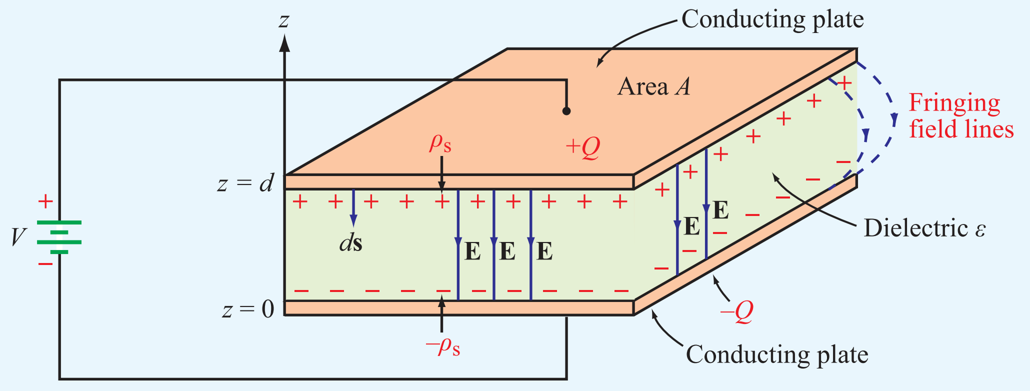

Figure 4-26 A dc voltage source connected to a parallel-plate capacitor (Example 4-14).

The charge density on the upper plate is $\rho_s = Q/A$. Hence, in the dielectric medium

$$\mathbf{E} = -\hat{\mathbf{z}}E,$$

and from Eq. (4.116), the magnitude of $\mathbf{E}$ at the conductor–dielectric boundary is $E = \rho_s/\varepsilon = Q/\varepsilon A$. From Eq. (4.118), the voltage difference is

$$V = -\int_0^d \mathbf{E} \cdot d\mathbf{l} = -\int_0^d (-\hat{\mathbf{z}}E) \cdot \hat{\mathbf{z}}\, dz = Ed, \qquad (4.122)$$

and the capacitance is

$$C = \frac{Q}{V} = \frac{Q}{Ed} = \frac{\varepsilon A}{d}, \qquad (4.123)$$

where use was made of the relation $E = Q/\varepsilon A$.

Example2-15: Capacitance per Unit Length of Coaxial Line

Obtain an expression for the capacitance of the coaxial line shown in Fig. 4-27.

Solution: For a given voltage V across the capacitor, charges $+Q$ and $-Q$ accumulate on the surfaces of the outer and inner conductors, respectively. We assume that these charges are uniformly distributed along the length and circumference of the conductors with surface charge density $\rho'_s = Q/2\pi bl$ on the outer conductor and $\rho''_s = -Q/2\pi al$ on the inner one. Ignoring fringing fields near the ends of the coaxial line, we can construct a cylindrical Gaussian surface in the dielectric in between the conductors with the radius r such that $a < r < b$. Symmetry implies that the $\mathbf{E}$-field is identical at all points on this surface, which is directed radially inward. From Gauss's law, it follows that the field magnitude equals the absolute

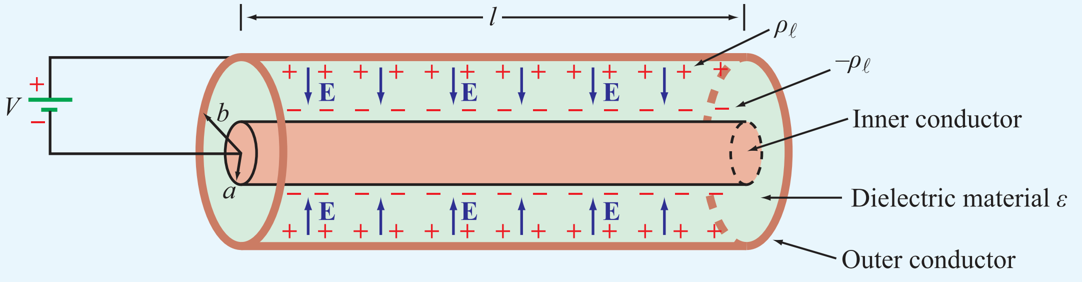

Figure 4-27 Coaxial capacitor filled with insulating material of permittivity ε (Example 4-15).

value of the total charge enclosed, divided by the surface area. That is,

$$\mathbf{E} = -\hat{\mathbf{r}} \, \frac{Q}{2\pi\varepsilon rl}. \tag{4.124}$$

The potential difference V between the outer and inner conductors is

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = -\int_a^b \left(-\hat{\mathbf{r}} \, \frac{Q}{2\pi\varepsilon rl}\right) \cdot (\hat{\mathbf{r}} \, dr)$$
$$= \frac{Q}{2\pi\varepsilon l} \ln\left(\frac{b}{a}\right). \tag{4.125}$$

The capacitance C is then given by

$$C = \frac{Q}{V} = \frac{2\pi\varepsilon l}{\ln(b/a)}, \tag{4.126}$$

and the capacitance per unit length of the coaxial line is

$$C' = \frac{C}{l} = \frac{2\pi\varepsilon}{\ln(b/a)} \qquad \text{(F/m)}. \tag{4.127}$$

Concept Question2-24: How is the capacitance of a two-conductor structure related to the resistance of the insulating material between the conductors?

Concept Question2-25: What are fringing fields and when may they be ignored?

2-10 Electrostatic Potential Energy

A source connected to a capacitor expends energy in charging up the capacitor. If the capacitor plates are made of a good conductor with effectively zero resistance, and if the dielectric separating the two plates has negligible conductivity, then no real current can flow through the dielectric, and no ohmic losses occur anywhere in the capacitor. Where then does the energy expended in charging up the capacitor go? The energy ends up getting stored in the dielectric medium in the form of ***electrostatic potential energy***. The amount of stored energy W_e is related to Q, C, and V.

Suppose we were to charge up a capacitor by ramping up the voltage across it from $\upsilon = 0$ to $\upsilon = V$. During the process, charge $+q$ accumulates on one conductor and $-q$ on the other. In effect, a charge q has been transferred from one of the conductors to the other. The voltage υ across the capacitor is related to q by

$$\upsilon = \frac{q}{C}. \tag{4.128}$$

From the definition of υ, the amount of work dW_e required to transfer an additional incremental charge dq from one conductor to the other is

$$dW_e = \upsilon \, dq = \frac{q}{C} \, dq. \tag{4.129}$$

If we transfer a total charge Q between the conductors of an initially uncharged capacitor, then the total amount of work performed is

$$W_e = \int_0^Q \frac{q}{C} \, dq = \frac{1}{2} \, \frac{Q^2}{C} \qquad \text{(J)}. \tag{4.130}$$

Using $C = Q/V$, where V is the final voltage, W_e also can be expressed as

$$W_e = \tfrac{1}{2} C V^2 \qquad \text{(J)}. \tag{4.131}$$

The capacitance of the parallel-plate capacitor discussed in Example 4-14 is given by Eq. (4.123) as $C = \varepsilon A/d$, where A is the surface area of each of its plates and d is the separation between them. Also, the voltage V across the capacitor is related to the magnitude of the electric field E in the dielectric by $V = Ed$. Using these two expressions in Eq. (4.131) gives

$$W_e = \tfrac{1}{2} \, \frac{\varepsilon A}{d} (Ed)^2 = \tfrac{1}{2} \, \varepsilon E^2 (Ad) = \tfrac{1}{2} \, \varepsilon E^2 \mathcal{V}, \tag{4.132}$$

where $\mathcal{V} = Ad$ is the volume of the capacitor. This expression affirms the assertion made at the beginning of this section, namely that the energy expended in charging up the capacitor is being stored in the electric field present in the dielectric material in between the two conductors.

The ***electrostatic energy density*** w_e is defined as the electrostatic potential energy W_e per unit volume:

$$w_e = \frac{W_e}{\mathcal{V}} = \frac{1}{2} \, \varepsilon E^2 \qquad \text{(J/m}^3\text{)}. \tag{4.133}$$

Even though this expression was derived for a parallel-plate capacitor, it is equally valid for any dielectric medium containing an electric field $\mathbf{E}$, including a vacuum. Furthermore, for any volume $\mathcal{V}$, the total electrostatic potential energy stored in it is

$$W_e = \frac{1}{2} \int_{\mathcal{V}} \varepsilon E^2 \, d\mathcal{V} \qquad \text{(J)}. \tag{4.134}$$

Returning to the parallel-plate capacitor, the oppositely charged plates are attracted to each other by an ***electrical force*** $\mathbf{F}_e$. In terms of the coordinate system of **Fig.2-28** , the

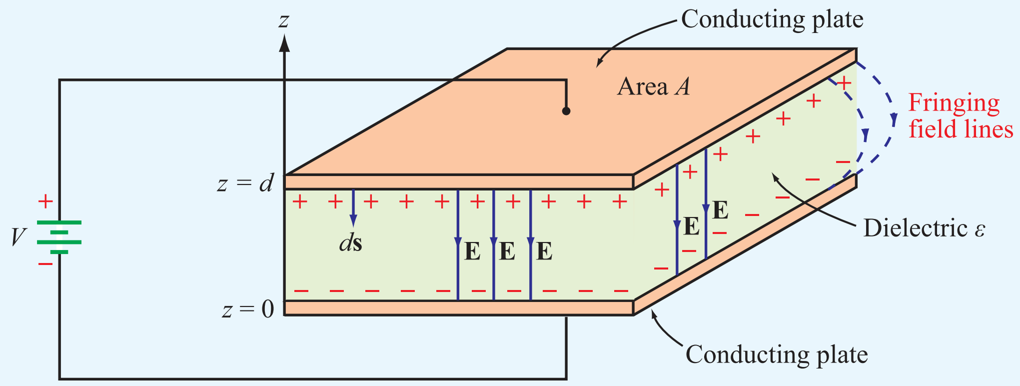

Figure2-28 A dc voltage source connected to a parallel-plate capacitor.

electrical force acting on the upper plate is along $-\hat{\mathbf{z}}$ (due to attraction by the lower plate). Hence, it is given by

$$\mathbf{F}_e = -\hat{\mathbf{z}}\, F_e \qquad \text{(force on upper plate).} \tag{4.135}$$

Our plan is to compute $\mathbf{F}_e$ from energy considerations. We start by converting the spacing d into a variable z and using $C = \varepsilon A/z$ in Eq. (4.131):

$$W_e = \tfrac{1}{2}\, CV^2 = \tfrac{1}{2}\, \frac{\varepsilon A V^2}{z}. \tag{4.136}$$

If V is maintained at a constant level, W_e decreases when increasing the separation z between the plates. If an external, upward-directed force $\mathbf{F} = -\mathbf{F}_e$ is applied to counter the electrostatic force $\mathbf{F}_e$ and used to move the upper plate upwards by a distance dz, the expended mechanical work is

$$dW = \mathbf{F} \cdot \hat{\mathbf{z}}\, dz. \tag{4.137}$$

The work dW is equal to the loss in electrostatic energy stored in the capacitor. That is,

$$dW = -dW_e. \tag{4.138}$$

Also, $\mathbf{F}_e = -\mathbf{F}$, which leads to

$$dW_e = \mathbf{F}_e \cdot \hat{\mathbf{z}}\, dz = -\hat{\mathbf{z}}\, F_e \cdot \hat{\mathbf{z}}\, dz = -F_e\, dz. \tag{4.139}$$

From Eq. (4.136),

$$dW_e = -\tfrac{1}{2}\, \varepsilon\, \frac{AV^2}{z^2}\, dz. \tag{4.140}$$

Equating Eqs. (4.139) and (4.140) and replacing z with d leads to

$$F_e = \tfrac{1}{2}\, \varepsilon\, \frac{AV^2}{d^2}, \tag{4.141a}$$

and

$$\mathbf{F}_e = -\hat{\mathbf{z}}\, \tfrac{1}{2}\, \varepsilon A\, \frac{V^2}{d^2} \qquad \text{(N).} \tag{4.141b}$$

(parallel-plate capacitor)

This is the electrostatic force exerted on the upper plate. The force on the lower plate is identical in magnitude and opposite in direction.

The relationship given by Eq. (4.139) pertains to a capacitor with $d\mathbf{l} = \hat{\mathbf{z}}\, dz$. We can generalize the result for $d\mathbf{l}$ along any direction as

$$\mathbf{F}_e = -\nabla W_e. \tag{4.142}$$

Example2-16: Force on Sliding Dielectric

The two plates of the parallel-plate capacitor shown in **Fig. 4-29** are each of length ℓ and width w, and the separation between them is d. The capacitor contains a dielectric block of dimensions $\ell \times w \times d$ and permittivity ε. The block can slide in and out of the capacitor cavity along its length dimension. Compute the force $\mathbf{F}_e$ acting on the dielectric block when it is partially outside of the cavity and the voltage across the capacitor is V.

Solution: From Eq. (4.122), the electric field inside the capacitor cavity is

$$E = \frac{V}{d}.$$

This is true in both the section containing the dielectric block and the section filled with air. The total electrostatic energy of

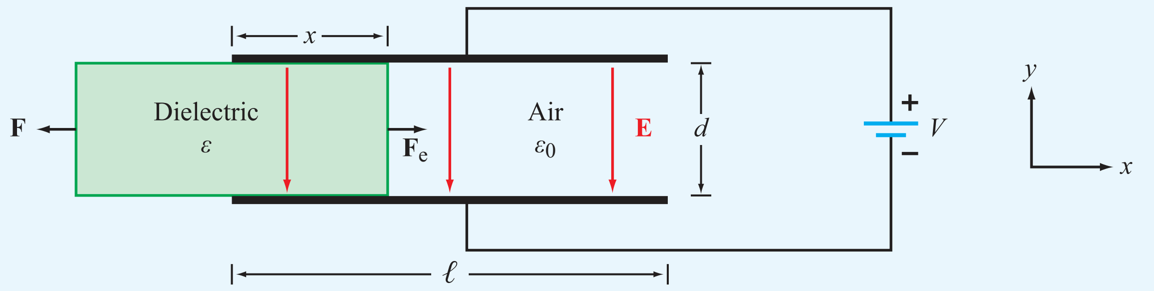

Figure 4-29 Parallel-plate capacitor with slidable dielectric block.

the capacitor consists of two components: one for the volume containing the dielectric block of permittivity ε and volume $\upsilon_1 = xwd$, and another for the volume containing air with ε_0 and volume $\upsilon_2 = (\ell - x)wd$. Hence,

$$\begin{aligned} W_e &= \tfrac{1}{2}\,\varepsilon E^2 \upsilon_1 + \tfrac{1}{2}\,\varepsilon_0 E^2 \upsilon_2 \\ &= \tfrac{1}{2}\,\varepsilon \left(\frac{V}{d}\right)^2 xwd + \tfrac{1}{2}\,\varepsilon_0 \left(\frac{V}{d}\right)^2 (\ell - x)wd \\ &= \tfrac{1}{2}\,\frac{V^2}{d}\,w[\varepsilon x + \varepsilon_0(\ell - x)]. \end{aligned} \tag{4.143}$$

Since $\varepsilon > \varepsilon_0$, the electrostatic energy is maximum when $x = \ell$ (dielectric block fully inside the cavity). Sliding the dielectric block out of the capacitor requires exerting an external mechanical force $\mathbf{F}$ to oppose the electrostatic force $\mathbf{F}_e$, whose tendency is to oppose reduction in W_e. Thus, the direction of $\mathbf{F}_e$ is to pull the block back into the capacitor.

The magnitude of $\mathbf{F}_e$ can be obtained from

$$\begin{aligned} F_e &= \frac{dW_e}{dx} \\ &= \frac{d}{dx}\left[\frac{1}{2}\,\frac{V^2}{d}\,w[\varepsilon x + \varepsilon_0(\ell - x)]\right] \\ &= \frac{1}{2}\,\frac{V^2}{d}\,w(\varepsilon - \varepsilon_0). \end{aligned} \tag{4.144}$$

Concept Question 4-26: To bring a charge q from infinity to a given point in space, a certain amount of work W is expended. Where does the energy corresponding to W go?

Concept Question 2-27: When a voltage source is connected across a capacitor, what is the direction of the electrical force acting on its two conducting surfaces?

Exercise 2-18: The radii of the inner and outer conductors of a coaxial cable are 2 cm and 5 cm, respectively, and the insulating material between them has a relative permittivity of 4. The charge density on the outer conductor is $\rho_\ell = 10^{-4}$ (C/m). Use the expression for $\mathbf{E}$ derived in Example 4-15 to calculate the total energy stored in a 20 cm length of the cable.

Answer: $W_e = 4.1$ J. (See EM.)

2-11 Image Method

Consider a point charge Q at a distance d above a horizontally infinite, perfectly conducting plate (**Fig. 4-30(a)**). We want to determine V and $\mathbf{E}$ at any point in the space above the plate, as well as the surface charge distribution on the plate. Three different methods for finding $\mathbf{E}$ have been introduced in this chapter The first method, based on Coulomb's law, requires knowledge of the magnitudes and locations of all the charges. In the present case, the charge Q induces an unknown and nonuniform distribution of charge on the plate. Hence, we cannot utilize Coulomb's method. The second method, based on Gauss's law, is equally difficult to use because it is not clear how to construct a Gaussian surface across which $\mathbf{E}$ is only tangential or only normal. The third method is based on evaluating the electric field using $\mathbf{E} = -\nabla V$ after solving Poisson's or Laplace's equation for V subject to the available boundary conditions, but it is mathematically involved.

Alternatively, the problem at hand can be solved using ***image theory***.

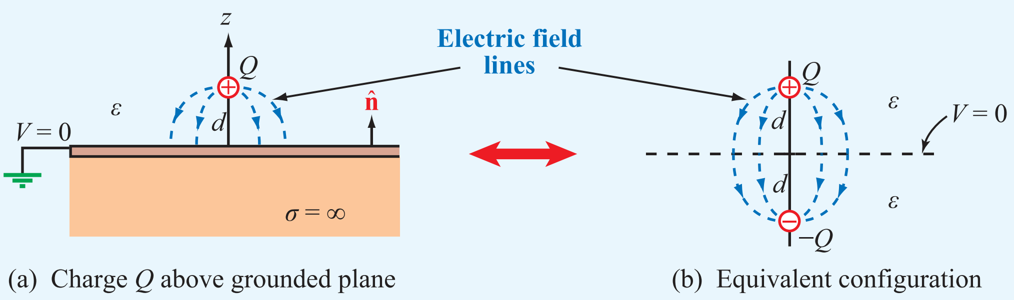

Figure2-30 By image theory, a charge Q above a grounded, perfectly conducting plane is equivalent to Q and its image $-Q$ with the grounded plane removed.

▶ Any given charge configuration above an infinite, perfectly conducting plane is electrically equivalent to the combination of the given charge configuration and its image configuration with the conducting plane removed. ◀

The image-method equivalent of the charge Q above a conducting plane is shown in the right-hand section of **Fig.2-30** . It consists of the charge Q itself and an image charge $-Q$ at a distance $2d$ from Q with nothing else between them. The electric field due to the two isolated charges can now be easily found at any point (x, y, z) by applying Coulomb's method, as demonstrated by Example 4-17. By symmetry, the combination of the two charges always produces a potential $V = 0$ at every point in the plane previously occupied by the conducting surface. If the charge resides in the presence of more than one grounded plane, it is necessary to establish its images relative to each of the planes and then to establish images of each of those images against the remaining planes. The process is continued until the condition $V = 0$ is satisfied everywhere on all grounded planes. The image method applies not only to point charges but also to distributions of charge, such as the line and volume distributions depicted in **Fig.2-31** . Once $\mathbf{E}$ has been determined, the charge induced on the plate can be found from

$$\rho_s = (\hat{\mathbf{n}} \cdot \mathbf{E})\varepsilon_0, \qquad (4.145)$$

where $\hat{\mathbf{n}}$ is the normal unit vector to the plate (**Fig.2-30(a)**).

Example2-17: Image Method for Charge above Conducting Plane

Use image theory to determine $\mathbf{E}$ at an arbitrary point $P = (x, y, z)$ in the region $z > 0$ due to a charge Q in free space at a distance d above a grounded conducting plate residing in the $z = 0$ plane.

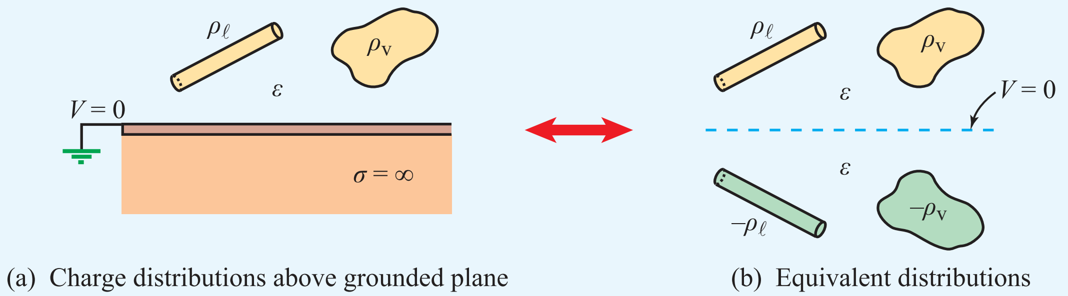

Figure2-31 Charge distributions above a conducting plane and their image-method equivalents.

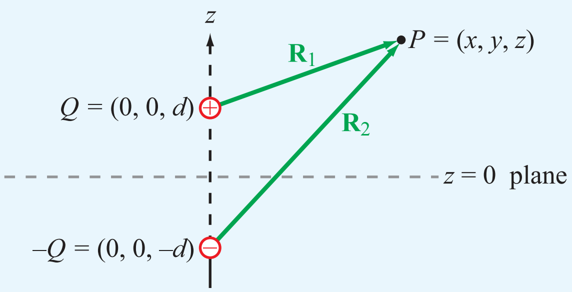

Figure2-32 Application of the image method for finding **E** at point P (Example 4-17).

Solution: In **Fig.2-32** , charge Q is at $(0,0,d)$, and its image $-Q$ is at $(0,0,-d)$. From Eq. (4.19), the electric field at point $P(x,y,z)$ due to the two charges is given by the following equation.

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q\mathbf{R}_1}{R_1^3} + \frac{-Q\mathbf{R}_2}{R_2^3}\right)$$
$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}y+\hat{\mathbf{z}}(z-d)}{[x^2+y^2+(z-d)^2]^{3/2}} - \frac{\hat{\mathbf{x}}x+\hat{\mathbf{y}}y+\hat{\mathbf{z}}(z+d)}{[x^2+y^2+(z+d)^2]^{3/2}}\right]$$

for $z \geq 0$.

Concept Question 2-28: What is the fundamental premise of the image method?

Concept Question 2-29: Given a charge distribution, what are the various approaches described in this chapter for computing the electric field **E** at a given point in space?

Exercise 2-19: Use the result of Example 4-17 to find the surface charge density ρ_s on the surface of the conducting plane.

Answer: $\rho_s = -Qd/[2\pi(x^2+y^2+d^2)^{3/2}]$. (See EM.)

Chapter 2 Summary

Concepts

- Maxwell's equations are the fundamental tenets of electromagnetic theory.
- Under static conditions, Maxwell's equations separate into two uncoupled pairs with one pair pertaining to electrostatics and the other to magnetostatics.
- Coulomb's law provides an explicit expression for the electric field due to a specified charge distribution.
- Gauss's law states that the total electric field flux through a closed surface is equal to the net charge enclosed by the surface.
- The electrostatic field **E** at a point is related to the electric potential V at that point by $\mathbf{E} = -\nabla V$ with V often being referenced to zero at infinity.
- Because most metals have conductivities on the order of 10^6 (S/m), they are treated in practice as perfect conductors. By the same token, insulators with conductivities smaller than 10^{-10} (S/m) often are treated as perfect dielectrics.
- Boundary conditions at the interface between two materials specify the relations between the normal and tangential components of **D**, **E**, and **J** in one of the materials to the corresponding components in the other.
- The capacitance of a two-conductor body and resistance of the medium between them can be computed from knowledge of the electric field in that medium.
- The electrostatic energy density stored in a dielectric medium is $w_e = \frac{1}{2}\varepsilon E^2$ (J/m^3).
- When a charge configuration exists above an infinite, perfectly conducting plane, the induced field **E** is the same as that due to the configuration itself and its image with the conducting plane removed.

Mathematical and Physical Models

Maxwell's Equations for Electrostatics

Name	Differential Form	Integral Form
Gauss's law	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$
Kirchhoff's law	$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$

Current density	$\mathbf{J} = \rho_v \mathbf{u}$
Poisson's equation	$\nabla^2 V = -\dfrac{\rho_v}{\varepsilon}$
Laplace's equation	$\nabla^2 V = 0$
Resistance	$R = \dfrac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}$
Boundary conditions	Table 4-3
Capacitance	$C = \dfrac{\int_S \varepsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}}$
RC relation	$RC = \dfrac{\varepsilon}{\sigma}$
Energy density	$w_e = \frac{1}{2}\varepsilon E^2$

Electric Field

Point charge	$\mathbf{E} = \hat{\mathbf{R}} \dfrac{q}{4\pi\varepsilon R^2}$
Many point charges	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \sum_{i=1}^{N} \dfrac{q_i(\mathbf{R} - \mathbf{R}_i)}{\lvert\mathbf{R} - \mathbf{R}_i\rvert^3}$
Volume distribution	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \int_{\mathcal{V}'} \hat{\mathbf{R}}' \dfrac{\rho_v \, d\mathcal{V}'}{R'^2}$
Surface distribution	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \int_{S'} \hat{\mathbf{R}}' \dfrac{\rho_s \, ds'}{R'^2}$
Line distribution	$\mathbf{E} = \dfrac{1}{4\pi\varepsilon} \int_{l'} \hat{\mathbf{R}}' \dfrac{\rho_\ell \, dl'}{R'^2}$
Infinite sheet of charge	$\mathbf{E} = \hat{\mathbf{z}} \dfrac{\rho_s}{2\varepsilon_0}$
Infinite line of charge	$\mathbf{E} = \dfrac{\mathbf{D}}{\varepsilon_0} = \hat{\mathbf{r}} \dfrac{D_r}{\varepsilon_0} = \hat{\mathbf{r}} \dfrac{\rho_\ell}{2\pi\varepsilon_0 r}$
Dipole	$\mathbf{E} = \dfrac{qd}{4\pi\varepsilon_0 R^3} (\hat{\mathbf{R}}\, 2\cos\theta + \hat{\boldsymbol{\theta}} \sin\theta)$
Relation to *V*	$\mathbf{E} = -\nabla V$

PROBLEMS

Sections 4-2: Charge and Current Distributions

*2-1 A cube 2 m on a side is located in the first octant in a Cartesian coordinate system with one of its corners at the origin. Find the total charge contained in the cube if the charge density is given by $\rho_v = xy^2 e^{-2z}$ (mC/m^3).

2.2 Find the total charge contained in a cylindrical volume defined by $r \le 2$ m and $0 \le z \le 3$ m if $\rho_v = 20rz$ (mC/m^3).

*2.3 Find the total charge contained in a round-top cone defined by $R \le 2$ m and $0 \le \theta \le \pi/4$ given that $\rho_v = 10R^2 \cos^2\theta$ (mC/m^3).

2.4 If the line charge density is given by $\rho_l = 24y^2$ (mC/m), find the total charge distributed on the y axis from $y = -5$ to $y = 5$.

Important Terms

Provide definitions or explain the meaning of the following terms:

boundary conditions
capacitance C
charge density
conductance G
conduction current
conductivity σ
conductor
conservative field
constitutive parameters
convection current
Coulomb's law
current density $\mathbf{J}$
dielectric breakdown voltage V_{br}
dielectric material
dielectric strength E_{ds}
dipole moment $\mathbf{p}$
electric dipole
electric field intensity $\mathbf{E}$
electric flux density $\mathbf{D}$
electric potential V
electric susceptibility χ_{e}
electron drift velocity $\mathbf{u}_{\mathrm{e}}$
electron mobility μ_{e}
electrostatic energy density w_{e}
electrostatic potential energy W_{e}
electrostatics
equipotential
Gaussian surface
Gauss's law
hole drift velocity $\mathbf{u}_{\mathrm{h}}$
hole mobility μ_{h}
homogeneous material
image method
isotropic material
Joule's law
Kirchhoff's voltage law
Laplace's equation
linear material
Ohm's law
perfect conductor
perfect dielectric
permittivity ε
Poisson's equation
polarization vector $\mathbf{P}$
relative permittivity ε_{r}
semiconductor
static condition
superconductor
volume, surface, and line charge densities

2.5 Find the total charge on a circular disk defined by $r \leq a$ and $z = 0$ if:

(a) $\rho_{\mathrm{s}} = \rho_{\mathrm{s0}} \cos\phi$ (C/m^2)

(b) $\rho_{\mathrm{s}} = \rho_{\mathrm{s0}} \sin^2\phi$ (C/m^2)

(c) $\rho_{\mathrm{s}} = \rho_{\mathrm{s0}} e^{-r}$ (C/m^2)

(d) $\rho_{\mathrm{s}} = \rho_{\mathrm{s0}} e^{-r} \sin^2\phi$ (C/m^2)

where ρ_{s0} is a constant.

2.6 If $\mathbf{J} = \hat{\mathbf{y}}4xz$ (A/m^2), find the current I flowing through a square with corners at $(0,0,0)$, $(2,0,0)$, $(2,0,2)$, and $(0,0,2)$.

***2.7** If $\mathbf{J} = \hat{\mathbf{R}}5/R$ (A/m^2), find I through the surface $R = 5$ m.

***2.8** A square with sides of 2 m has a charge of 40 μC at each of its four corners. Determine the electric field at a point 5 m above the center of the square.

2-9 Three point charges, each with $q = 3$ nC, are located at the corners of a triangle in the x–y plane, with one corner at the origin, another at $(2\text{ cm}, 0, 0)$, and the third at $(0, 2\text{ cm}, 0)$. Find the force acting on the charge located at the origin.

***2.10** Charge $q_1 = 6$ μC is located at $(1\text{ cm}, 1\text{ cm}, 0)$ and charge q_2 is located at $(0, 0, 4\text{ cm})$. What should q_2 be so that $\mathbf{E}$ at $(0, 2\text{ cm}, 0)$ has no y component?

2.11 A line of charge with uniform density $\rho_\ell = 8$ (μC/m) exists in air along the z axis between $z = 0$ and $z = 5$ cm. Find $\mathbf{E}$ at (0,10 cm,0).

***2-12** The electric flux density inside a dielectric sphere of radius a centered at the origin is given by

$$\mathbf{D} = \hat{\mathbf{R}}\rho_0 R \qquad (\mathrm{C/m^2})$$

where ρ_0 is a constant. Find the total charge inside the sphere.

2.13 In a certain region of space, the charge density is given in cylindrical coordinates by the function

$$\rho_{\mathrm{v}} = 5re^{-r} \quad (\mathrm{C/m^3}).$$